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ELEMENTS

OF

ARITHMETIC, ALGEBRA, AND GEOMETRY.

ELEMENTS

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Printed by John Stark.

ELEMENTS

V. 1827.

OF

ARITHMETIC, ALGEBRA, AND GEOMETRY,

FOR THE

USE OF THE STUDENTS

IN THE

Edinburgh School of Arts.

BY GEORGE LEES, A. M.

TEACHER OF MATHEMATICS IN THE SCOTTISH MILITARY AND
NAVAL ACADEMY.

EDINBURGH:
PRINTED FOR ADAM BLACK AND WILLIAM TAIT;
AND LONGMAN & CO. AND CHARLES TAIT,
LONDON.

1826.

553.

THIS VOLUME

IS

MOST RESPECTFULLY DEDICATED

TO

LEONARD HORNER, Esq. F.R.SS.L.&E.

FOUNDER AND SECRETARY

OF THE

EDINBURGH SCHOOL OF ARTS,

BY

HIS OBLIGED AND OBEDIENT SERVANT,

GEORGE LEES.

PREFACE.

THE object of the present publication, is to supply the students of the Edinburgh School of Arts, and those of similar Institutions, with a condensed yet comprehensive course of Mathematics. It will be readily acknowledged, by those who know any thing of Mathematical Science, that it, of all subjects whatever, can least be taught in the form of lecture. There is through the whole such a concatenation of dependent truths, that unless the Student secure these firmly as he proceeds, he will soon find himself in the same hopeless predicament, as if endeavouring to ascend an eminence by means of a ladder divested of steps. Nor would we insinuate from this, that the study of Mathematics is difficult; on the contrary, we believe that none is more easy or better fitted for a rational mind. From a few axioms or intuitive truths, which are no sooner enunciated than believed, the Mathematician raises the most superb structure, from which he can contemplate, as with a telescope, the beauties of the scientific landscape. Treasures which lay hid before, are gradually opening to his view, and pressing themselves upon his acceptance. It is not with Mathematics, as with the other sciences, where one theory

TENTS.

RT I.

METIC.

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553.

Page
Addition of Fractions 76
Subtraction of Fractions 77
Multiplication of Fractions . . . . . . ib.
Division of Fractions 78
Involution and Evolution 79
Surds 100
On the Reduction of Surds . . . . ib.
Addition and Subtraction of Surds 103
Multiplication and Division of Surds 104
Involution and Evolution of Surds 105
Simple Equations 106
Quadratic Equations 121
On Ratios and Proportion 131
On Proportion 134
On Arithmetical Progression 139
On Geometrical Progression 141

ELEMENTS OF GEOMETRY.

Book I. 145
Book II. 167
Book III. 183
Book IV. 193

INTRODUCTION.

THE word MATHEMATICS, like most other of our scientific terms, is of Grecian origin, and seems originally to have signified knowledge, or learning in general. It is now, however, greatly limited in its signification, being applied to designate that science exclusively which treats of quantity.

Quantity may be contemplated under two different forms; either as made up of separate and distinct parts, or as one extended and continuous whole. When quantity is considered as a collection of separate and distinct parts, or as an aggregate of several things of the same kind, it is called *Number*; and that portion of Mathematics which proposes to investigate the power and properties of number is denominated ARITHMETIC.

When quantity is considered as one extended and continuous whole, such as in the case of a line, a surface, or a solid, it is called extension; and that division of Mathematics, which proposes to investigate the properties of figured extension, is denominated GEOMETRY.

Algebra, another very important branch of Mathematics, seems to hold a kind of middle ground between Arithmetic and Geometry; possessing several of the distinctive qualities of each, yet the exclusive characteristic of neither. For, although it is closely allied to the former, in regard to the manner of its operations, yet it is equally so to the latter, in regard to the generality of its conclusions. Hence, it has been sometimes very appropriately called UNIVERSAL ARITHMETIC. It serves as that link by which Geometry is connected with Arithmetic; or that medium through which Geometrical relation may be exhibited in number.

When quantity is considered in the abstract, *that is*, apart from all considerations of its existing in any material object whatever, the science is called PURE MATHEMATICS. When quantity is considered as not abstracted from the subject of its existence, the science is called MIXED MATHEMATICS. That quantity cannot exist, unless in some Physical object, is certainly very evident; it is nevertheless convenient and philosophical, by a process of mental abstraction to *conceive of it separately*. By this means we become armed with a series of speculative truths, which enable us to push our way through new and untried difficulties; to overcome every obstruction, however formidable, by which truth may be encompassed, and to proceed with a firm and a sure step in all our scientific investigations.

The study of Mathematics, whether as to their elegance or utility, cannot be better recommended than in the words of Dr Barrow, in his inaugural oration when appointed professor of Mathematics at Cambridge. "The Mathematics, he observes, effectually exercise, not vainly delude, nor vexatiously torment studious minds with obscure subtleties, but plainly demonstrate every thing within their reach, draw certain conclusions, instruct by profitable rules, and unfold pleasant questions. These disciplines likewise inure and corroborate the mind to a constant diligence in study; they wholly deliver us from a credulous simplicity, most strongly fortify us against the vanity of scepticism, effectually restrain us from a rash presumption, most easily incline us to a due assent, and perfectly subject us to the government of right reason. While the mind is abstracted and elevated from sensible matter, distinctly views pure forms, conceives the beauty of ideas, and investigates the harmony of proportions, the manners themselves are sensibly corrected and improved,—the affections composed and rectified,—the fancy calmed and settled,—and the understanding raised and excited to more divine contemplations."

The fundamental principles of mathematics are so few and so easily apprehended—the chain of reasoning so complete and satisfactory—the truths established so clear and perspicuous—the relations discovered so replete with elegance and beauty—that, whilst the reasoning faculties are insensibly gaining strength by the most delightful exercise, the mind itself becomes more and more gratified by the clearness of its views, by the certainty of its conclusions, and by the justness and distinctness of its conceptions. But, besides that mathematics may thus be considered as one of the best exercises to the understanding, they merit our most scrupulous attention, even from equally important considerations; inasmuch as they lie at the very basis of our finest commercial speculations; are intimately connected with the various branches of Mechanical Philosophy; with almost all the great sources of national wealth and power; and with many of those arts which contribute to the elegance and comfort of social life.

PART I.

ARITHMETIC.

1. Any single object whatever is called a unit or *one*. If to this unit another be added, these together form the number called *two;* which increased in like manner by another unit, gives the next greater number denominated *three.* By continuing to add unity in this way to the number already obtained, the successive numbers *four*, *five*, *six*, &c. are formed. From this method of conceiving the formation of number, it is obvious that there is no limit to its magnitude; for, however great any number may be, it may still be augmented by a unit, or by any number of units.

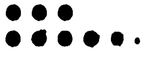
2. In the early stages of society, when words, the signs of ideas, were but few, mankind would naturally represent any small number by some palpable symbols, such as *nuts*, *pebbles*, or *shells* arranged in a row. It would be found, however, when the objects to be represented were numerous, that this method of arranging them in a *single* row would only convey a very confused idea of multitude. In order, therefore, to greater precision, they would subsequently form them into two rows, or count them off by pairs. Hence the origin of the dual number, and of the terms *brace*, *couple*, &c. *Thirty-one*, for example, might be represented by as many small shells or pebbles, which, when ranged into successive pairs, would amount to fifteen pairs and one over, thus:—

· · · · · · · · · · · · · · ·
· · · · · · · · · · · · · · · ·

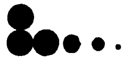
If a pebble or shell of double the size were now taken to represent each pair, then the number, by the same process of arrangement, would be represented by seven pairs of these double-sized counters and one over, together with one counter of the first size, thus:—

● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ·

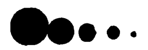
Again, if counters of double the size of these last be taken, the number will be represented by three pairs of this third size of counters and one over; one of the second size, together with one of the first, thus:—



In order to farther condensation, if each pair were represented, as before, by a counter still larger, then the whole number would be represented by three counters of the fourth size, one of the third, one of the second, and one of the first, thus:—



Lastly, If a counter be taken equal to two of this fourth size, then the number, *thirty-one*, will be represented by one counter of the fifth size, one of the fourth, one of the third, one of the second, and one of the first, and would stand thus:—



Such are the steps by which we may conceive a savage, prompted by curiosity, to have proceeded in forming a condensed representation of any number of similar objects.

This mode of classifying numbers, by distributing them into successive pairs, is called the *Binary scale*, from the circumstance of its *root* or *index* being *two*.

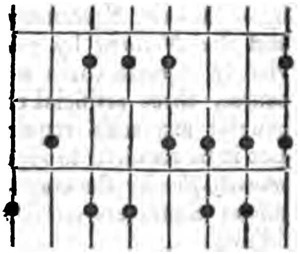
3. The inconvenience or utter impossibility of procuring natural objects of the necessary gradation of size, would naturally suggest the idea of affixing a certain gradation of value to rank, independent of magnitude. This important link in the chain of improvement was made by the Greeks at a very early period, and by them communicated to the Romans, who continued, during their whole career of empire, to practise a kind of tangible Arithmetic. In this, it is evident, that the augmenting value of rank would be quite arbitrary, depending in every case on a key, to be fixed upon by convention. If it be required, for example, to represent the number *thirty-one*, it being agreed that the value of each successive counter, reckoning from right to left, is double of that which precedes it, it will stand thus:

e d c b a
● ● ● ● ●

Where the counter *a* will represent one unit, *b* two units, *c* four units, *d* eight units, and *e* sixteen units, in all thirty-one units.

4. The general method of representing *any number* on the Binary scale will be readily understood from a few words of explanation. Let there be a series of parallel bars, and suppose that it is required to represent the number *fifty-seven*. Instead of placing the *fifty-seven* counters on the first bar, it will amount to the same thing to place *twenty-eight* on the second, and *one* on the first, since each of the counters on the

second bar represents two units. In the same manner *twenty-eight* on the second bar will be represented in effect by *fourteen* on the third, or by *seven* on the fourth. Again, *seven* on the fourth bar will be the same as *three* on the fifth and *one* on the fourth. But *three* on the fifth will be equivalent to *one* on the sixth and *one* on the fifth. Hence the representation of *fifty-seven* on the *Binary scale* will be effected by *one* counter on the first bar, *one* on the fourth, *one* on the fifth, and *one* on the sixth, thus:



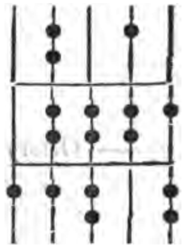
By a similar process of decomposition *seventy-nine* would be represented thus:

One hundred and *eighty-two* thus:

To represent any number on this scale, it is obvious that one counter for each bar is sufficient; because, instead of having two counters on any one bar, we may place one counter on the next higher.

5. Similar to the *Binary* is the *Ternary* scale of numeration, which reckons by successive *threes* or *triads*. On this scale a counter on any bar is equivalent to three counters on the bar immediately below it.

Let it be required to represent the number *fifty-seven* on this scale. In place of having *fifty-seven* counters on the first bar, we may have *nineteen* on the second, since nineteen threes or triads are exactly fifty-seven units. Again, *nineteen* counters on the second bar are equivalent to *six* on the third, or *two* on the fourth bar, and *one* on the second; *fifty-seven* will therefore be represented on the *Ternary scale*, by *one* counter on the second bar, and *two* on the fourth, thus:



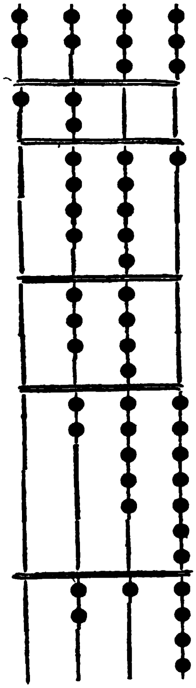
Seventy-nine would be represented thus:

One hundred and *twenty-eight* thus:

In this scale it is evident that two sets of counters are necessary; or, in figurative Arithmetic, two artificial characters. Let the characters 1 and 2 respectively denote *one* counter and *two* counters, and let the place of a bar on which no counter is placed be mark-ed by the cipher (0); then any number may be easily expressed on this scale.

Fifty-seven would be expressed thus: 2010
Seventy-nine thus: - 2221
One hundred and *twenty-eight* thus: 11202

6. From what has been stated, it will be easy to represent any number, on any scale whatever. The *Quaternary* scale proceeds by *fours*—the *Quinary* by *fives*—the *Senary* by *sixes*—the *Septenary* by *sevens*—the *Octary* by *eights*—and the *Nonary* by *nines*. The *Quaternary scale* obviously requires three sets of counters for each bar, and, of course, three artificial characters—the *Quinary five*, or in general any scale requires one less than its index or root. Let it be required to represent the number *one hundred* and *seventy-five* by the counters, and also to express it by the artificial characters on these various scales. It will stand as follows:

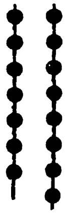


	By Counters.	By the Artificial Characters.
On the Quaternary .		2233
——— Quinary .		1200
——— Senary . .		451
——— Septenary .		340
——— Octary . .		257
——— Nonary . .		214

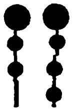
7. The Denary scale, which advances by tens, seems to have prevailed more generally, obviously from the familiar practice of counting by the fingers on both hands. This scale will evidently require nine sets of counters, and nine artificial characters.

Seventy-eight on the *Denary* scale would stand thus:

And by the artificial characters, thus: 78.



As the increased number of counters necessary for this scale might be inconvenient, an advantage will be gained by employing a counter of a larger size, to denote half the index of the scale, or five. *Seventy-eight* would then be represented in the abridged form thus:



8. The Denary scale was employed by the ancient Greeks and Romans, who performed their calculations by counters, on a board called an *Abax* or *Abacus*. A series of parallel grooves represented the bars, and the counters were originally pebbles, or small white stones, called *Calculi;* from which arose the verb *calculare,* and hence also our English term *to calculate.*

The Chinese use a board called a *Swanpan,* on which they perform calculations with astonishing rapidity. The *swanpan,* although a little dissimilar to the *abacus* in construction, is the same in principle. In it, the parallel grooves stretched transversely or across the board, from right to left, whereas in the *abacus* they run longitudinally, or up and down.

This method of reckoning by palpable symbols seems to have been almost universally prevalent throughout Europe until the fifteenth century, when it began to give place to the more convenient and expeditious mode, by written characters. There are, however, several places in Russia where this tangible arithmetic still prevails, as the only means of calculation. In Shakespeare's comedy of the *Winter's Tale,* written at the commencement of the seventeenth century, the clown, staggered at a simple multiplication, exclaims, "that he must try it with counters."

9. Although any number whatever might have been employed as the index or root of the arithmetical scale, yet the selection is not a matter of mere indifference. If the root were smaller, a less number of *distinct* characters would be sufficient to express any number; but these would require to be oftener repeated. Thus, the Binary scale only requires *one* written character expressive of number, whereas the Denary requires *nine*. To express the same number, however, on these two scales, more characters will be required by the Binary than by the Denary, as will be evident by expressing any number, suppose *eight hundred* and *forty-five*, on both.

By the Binary it will stand thus: 1101001101; and by the Denary thus: 845.

The Denary scale, which was generally used by the ancients, and now universally by the moderns, is perhaps better fitted for arithmetical computations than any other of those scales which have been mentioned, but is not the best that might have been adopted. The index or root *ten* of the Denary scale has only two aliquot parts, which are *two* and *five*; in other words, *ten* is only divisible by the numbers *two* and *five*, without any remainder. Had the *Duodenary* scale been adopted, whose index, *twelve*, is divisible by more numbers than that of the Denary, viz. by *two*, *three*, *four*, or *six*, a number expressed on the *Duodenary* scale would have been divisible by more divisors than if it had been expressed on the Denary; and hence fewer fractions would have occurred from the division of numbers expressed on the former scale than do from those expressed on the latter. It is a curious fact, that the celebrated Charles XII. of Sweden, a very short time before his death, seriously deliberated on a scheme of introducing this system of notation into his dominions. He seems also, at another time, to have had a strong desire of changing the index of the arithmetical scale from *ten* to *sixty-four*, because 64 is both a square and a cube number, and when continually divided by 2, is at last reduced to unity. "This idea," says Voltaire, "only shows that he delighted in every thing extraordinary and difficult."

10. Before taking leave of the subject of Arithmetical Scales, it seems proper to explain the method of transferring any numerical expression from one scale to another.

When a number is expressed by any other scale, let it be required to express it on the Denary or common scale. According to the principle of notation, the value of any figure is increased as many times as there are units in the index of the scale, by each removal of place toward the left hand; hence the method is obvious.

Ex. Transfer the number 2342 from the Quinary to the Denary scale.

$$\begin{array}{r}
2 = 2 \\
5\times4 = 20 \\
5\times5\times3 = 75 \\
5\times5\times5\times2 = 250 \\
\hline
\end{array}$$

Ans. 347 on the Denary or common scale.

Ex. 2. Express by figures on the Denary scale the number 7643 on the Octary.

$$\begin{array}{r}
3 = 3 \\
8\times4 = 32 \\
8\times8\times6 = 384 \\
8\times8\times8\times7 = 3584 \\
\hline
\end{array}$$

Ans. 4003 on the Denary.

Again, when a number is given on the Denary scale, let it be required to express it on any other scale. The method of doing this will be readily understood from an example.

Ex. 1. Transfer the number 8576 from the *Denary* to the *Senary* scale.

$$\begin{array}{r|l}
6)\underline{8576} & \\
6)\underline{1429} & 2 \\
6)\underline{238} & 1 \\
6)\underline{39} & 1 \\
6)\underline{6} & 3 \\
1 &
\end{array}$$

Ans. 13112.

Ex. 2. Transfer the number 105463 from the Denary to the Quinary scale.

$$\begin{array}{r|l}
5)\underline{105463} & \\
5)\underline{21092} & 3 \\
5)\underline{4218} & 2 \\
5)\underline{843} & 3 \\
5)\underline{168} & 3 \\
5)\underline{33} & 3 \\
5)\underline{6} & 3 \\
1 & 1
\end{array}$$

Ans. 11333323.

Of the Grecian, Roman, and Modern Notation by Written Characters.

11. The Greeks expressed numbers by means of ***twenty-seven*** distinct characters. Twenty-four of these were furnished by the letters of their alphabet, to which they added three additional ones to make up the number. The first nine letters denoted ***units***, or counters on the first bar; the second nine, ***tens***, or counters on the second bar; and the third nine, ***hundreds***, or counters on the third bar. By writing a small dash below or above these characters, their value was increased a thousand or ten thousand times.

12. The principal characters which the Romans employed in their notation, and which are still in use for representing dates, and numbering chapters, &c. were I, V, X, L, C, D, M, by the various combinations of which, they were enabled to express any number.

Probable origin of these characters. The most convenient, and even the most natural way of representing unity, by a written character, is by means of a stroke, I; two, in like manner, by two strokes, II; three by three strokes, III; and so on, till the reckoner reached the number ten, when in order to intimate that the first bar, or that of units, was completed, he would throw a dash across the mark for unity, and thus might arise X, the mark for ten. Were this separated into two, from the point where the strokes intersect either of them, V would become the mark for the half of ten, or for five. When the reckoner arrived at the third bar, or that of hundreds, he would combine three strokes, E, which, from greater dispatch in writing, would, in process of time, assume a round form, C. This character, E, separated into two, would give Γ or L, either of which L would represent fifty. In like manner, when he arrived at the fourth bar, or that of thousands, he would combine four strokes, M, which assuming a round form, would appear thus, ↀ, or CIƆ. This last, separated into two parts, would give CI or IƆ, either of which IƆ being more condensed into the form D, would signify the half of a thousand, or five hundred. The characters, CIƆ, and its half, IƆ, are sometimes used without any contraction; the former of which is increased ten times by writing Ɔ on the right hand and C on the left; the latter ten times by writing Ɔ on the right hand merely. Thus in regard to the increasing of CIƆ,

CIƆ=1000 and CCIƆƆ=10,000. Next in respect of the character IƆ, IↃ=500 and IↃↃ=5000. Farther, a stroke above any of these characters, increases its value a thousand times.

Thus M=1000, and $\overline{\text{M}}$=1,000,000.

The rules by which the Romans expressed any number by those characters, whose origin we have endeavoured to trace, are the following: First, When the same character is repeated any number of times, its value is to be taken as often.

Second, When a character of less value is placed on the left hand of one of greater value, the value is to be deducted from that of the greater.

Lastly, When a character of less value is placed to the right of one of greater value, its value is to be added to that of the greater.

Thus X=10, XX=20, and XXX=30.
IV=4, IX=9, and XIX= 19.
VI=6, XI=11, and LI=51.

13. The modern notation, which is far superior to that of either the Greeks or Romans, uses nine artificial characters expressive of number, as has been already alluded to, together with the cipher, which of itself implies a privation of value. These characters are, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, which, with the exception of the cipher, are called digits, (a term derived from the Latin word *digitus*, signifying the finger,) in reference to the primitive and predominating custom of reckoning by the fingers. They were first introduced into Europe by the Moors, who are said to have borrowed them from the Indians. Hence, the *modern* or *common* notation is sometimes called the *Indian*.

14. In order to facilitate the writing or reading of numbers, the digits, by which they are expressed, are sometimes classed into periods of six figures each, beginning from the units place. To these periods distinct names are given, which are as follows: *units*, *millions*, *billions*, *trillions*, *quadrillions*, *quintillions*, *sextillions*, &c. These periods are farther divided into half periods of three figures each. The first half period is called the period of units, the second that of thousands throughout all the full periods. Thus the number

$\overbrace{5,482,}^{\text{Billions.}}\overbrace{584,878,}^{\text{Millions.}}\overbrace{284,800}^{\text{Units.}}$ which expresses the number of square feet in the surface of our globe, is read 5 thousand 482 *billions*, 584 thousand 878 *millions*, 284 thousand 800 *units*. The French, in a manner somewhat simpler, divide numbers into periods of three figures each, which are called, respectively, *units*, *thousands*, *millions*, *billions*, *trillions*, &c. increasing in a thousand fold proportion. Hence, the French and English modes of numeration agree as far as hundreds of millions, but the English billion is a thousand times greater than the French.

15. It is sometimes convenient to conceive the digits by which any number is expressed, as denoting *units* of various degrees of value. By the palpable symbols, for example, on the Denary scale, *ten* counters on the first bar are equivalent to *one* on the second, and *ten* on the second equivalent to *one* on the third &c. So in figurate arithmetic, ten units of the *first order* are equivalent to one of the *second*, and ten of the second equivalent, in like manner, to one of the *third*, &c. In this way, simple units are called units of the *first order*; tens, units of the *second order*; hundreds, units of the *third order*; and so on.

16. However numerous the several operations to which numbers are subjected may appear, still they are all reducible to those of conjoining and separating. When two or more numbers are conjoined, that is, formed or collected into one, which is called their *sum*, the process by which this is effected is denominated *Addition*. When one number is drawn, or separated from another, so as to leave their *difference*, the process by which this is accomplished is called *Subtraction*. If the numbers to be added are all equal to one another, the process is capable of being much abridged, and then assumes the name of *Multiplication*. If the Subtraction be limited to the continued withdrawing of the same quantity from another, it is then also capable of abbreviation, and takes the name of *Division*.

ADDITION.

17. Addition, according to what is stated in the preceding article, is the method of finding a number which shall be equal to two or more given numbers.

The addition of numbers may be effected by setting out from one of them, and counting forwards as many units as are contained in each of the others successively. Suppose it were required, for example, to find the sum of 7 units, 5 units, and 6 units. By adding unity *five* times to 7, we obtain in succession the numbers *eight*, *nine*, *ten*, *eleven*, and, lastly, *twelve*. Again, by adding unity *six* times to twelve, we arrive in like manner at *eighteen*, thus showing that the sum of the given numbers is 18. Were the numbers large, this method would evidently be insupportably tedious, burdensome to the memory, and also much exposed to error. We have, therefore, recourse to another, founded on the principle of notation, and that the sum of several wholes is equal to the sums of their several parts.

Ex. 1. Required the sum of 5683 and 4213.

Arranging these numbers, so that units of the same order may stand under one another; we have 3 units and 3 units, which are together equal to 6 units; 1 ten and 8 tens, equal to 9 tens; 2 hundreds and 6 hundreds, equal to 8 hundreds; 4 thousands and 5 thousands, equal to 9 thousands. Hence the sum of the two given numbers is 9 *thousands*, 8 *hundreds*, 9 *tens*, and 6 *units*.

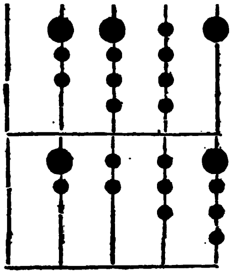
5683
4213
———
9896

Ex. 2. Required the sum of 7845, 6238, 507, and 59.

The sum of the units of the first order is 29, which (No. 15.) are equal to 2 units of the second order, and 9 of the first. Writing these 9 units of the first order, and carrying the 2 units of the second order to the next column, we there find the sum of this second order of units to be 14, which are equal to 4 units of the second order, and 1 of the third. Writing the 4, and adding the 1 to the sum of the units in the next column, or that of the third order of units, we obtain 16, which, in like manner, is equal to 6 units of the third order, and 1 of the fourth. This unit of the fourth order being added to the sum of the units in its proper column, gives 14, which is equal to 4 units of the fourth order, and 1 of the fifth. In the same way may the process be continued throughout any number of columns. Hence the sum of the given numbers is expressed by *one* unit of the fifth order, *four* of the fourth, *six* of the third, *four* of the second, and *nine* of the first.

7845
6238
507
59
———
14649

Same example by the *Abacus*, where the large counter denotes 5, or half the index of the scale.



The sum of the counters on the first bar is *twenty-nine*, which are equivalent to *nine* on the first, and *two* on the second. The sum of the counters on the second bar is already *twelve*, which, together with the two carried, makes *fourteen:* these, again, been diminished by *one*, *three* will remain; from which, if the *one* in the subtrahend be taken, *two* will be left. Again, as the units of the third order in the subtrahend, exceed those in the minuend, we, in like manner, in order to effect the subtraction, take *one* unit from the *nine* of the fourth order, which unit, being equal to *ten* of the third order, will, with the *three*, make *thirteen* units of this order; now, if from these the *five* units of the subtrahend be taken, there will remain *eight*. Lastly, in consequence of *one* unit being taken from the *nine* of the fourth order, *eight* will remain, from which, if *six* be taken, *two* will be left. In the same way may the process be continued throughout any number of columns.

19. This mode of procedure, which is certainly the most natural, is adopted by the Continental Arithmeticians, but differs a little from that employed in Britain. To take the same example as before. Here (in order to effect the subtraction) we add *ten* units of the first order to the *five* of the minuend, and from this sum deduct *eight*, and *seven* remains. Now, in order that the difference may remain the same, we must also increase the subtrahend by *ten* units of the first order, or, which will amount to the same thing, by *one* of the second. This being done, we shall have *two* units of the second order to subtract from *four*, which will leave a remainder of *two*. Adding *ten* units of the third order, in like manner, to the *three*, we obtain thirteen, from which, if the *five* in the subtrahend be taken, there will remain *eight*. Again, as the minuend was increased by *ten* units of the third order, we must also increase the subtrahend by *ten* units of this order, or, which is equivalent, by *one* unit of the fourth order. The units of the fourth order being increased by this *one*, will become *seven*, which, subtracted from *nine*, leave *two* as a remainder.

$$\begin{array}{r} 9345 \\ 6518 \\ \hline 2827 \end{array}$$

This process explains the meaning of the terms *borrowing* and *carrying*, as used in arithmetic. The *one* we carry is always equal to the *ten* we borrow, because it occupies the next higher place on the scale.

20. Subtraction may be proved by itself, or by the help of addition. For it is obvious, that, if the difference as obtained be correct, it must, when subtracted from the greater, leave the less as a remainder; or, when added to the less, reproduce the greater. Still, however, the proof is not absolutely certain, *that is*, it is not demonstration, but merely affords presumptive evidence of the operation being right.

MULTIPLICATION.

21. Multiplication, as already stated (No. 16), is nothing else than a compendious method of performing addition, when the numbers to be added are all equal to one another. Hence it may be defined to be the method of finding the amount of a given number, repeated a certain number of times.

This given number is called the *multiplicand;* the number of times that it is repeated, the *multiplier;* and the result, the *product.* The multiplicand and multiplier are also sometimes denominated *factors.*

When the multiplier is small, the product may be obtained by actually performing the addition. Required, for example, the amount of *seven* taken *four* times. Here we have only to write the multiplicand, seven, as often as there are units in the multiplier, four, and then add the four *sevens* together, thus:

$$\begin{array}{r}7\\7\\7\\7\\\hline 28\end{array}$$

In this way the products of every two digits may be found, which, when tabulated and committed to memory, will greatly facilitate the process of multiplication. A table of this description, called, from its use, the ***Multiplication Table,*** was first formed by Pythagoras, a celebrated Grecian philosopher, in honour of whom, it is sometimes called the ***Pythagorean Table.*** With a little extension, it is as follows:

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

The product of any two numbers which the table contains, will be found at the junction of the vertical and horizontal columns, at whose extremities the two numbers respectively appear. Thus, to find the product of 6 and 9, we have only to observe, where the columns, at whose extremities 6 and 9 are respectively written, meet one another, and there we find the product 54.

EXAMPLES IN MULTIPLICATION.

Ex. 1.	Ex. 2.
2321	7854
3	6
6963	47124

The process in Ex. 1. is sufficiently obvious. In Ex. 2. six times four units are twenty-four units, equal to two units of the second order, and four of the first; six times five are thirty, which, with the two carried, are thirty-two, equal to three units of the third order, and two of the second; six times eight are forty-eight, which, with the three carried, make fifty-one, equal to five units of the fourth order, and one of the third; six times seven are forty-two, which, with the five carried, are forty-seven, equal to four units of the fifth order, and seven of the fourth.

22. To multiply any number by 10, it is merely necessary to annex a cipher to it. For, by doing this, each collection of its units is increased ten times; *that is*, the units are changed into tens, the tens into hundreds, and the hundreds into thousands, &c. or the whole number is rendered ten times as great as at first.

In the same manner, to multiply any number by 100, it is only necessary to annex two ciphers to it. For by annexing one cipher, its value (No. 22) is increased ten times, and, by annexing another cipher, this last value will also be increased ten times; *that is*, the value, after two ciphers are annexed, will be ten times ten times, or a hundred times as great as at first. Hence we draw this general conclusion, that when the multiplier is expressed by unity with any number of ciphers annexed, it is merely necessary to annex that number of ciphers to the multiplicand, in order to obtain the product.

Ex. To multiply 75 by 10, we must annex one cipher, and the product will be 750. To multiply it by 100 we must annex two ciphers, and the product will be 7500, and so on.

23. If the digit of the multiplier be any other than unity,

the product will be obtained by multiplying by this digit, and then annexing the cipher.

Thus, for example, the product of 752 multiplied by 400, will be found, by multiplying 752 by 4, and then annexing two ciphers to this product. This is obvious, for, when 752 is multiplied by 4, its value is, of course, increased four times; and when, to this quadruple sum, two ciphers are annexed, *its* value is increased a hundred times. Hence this last value will be a hundred times four times, or four hundred times the first; *that is*, (Def.) it will be the product of 752, multiplied by 400.

752
400

300800

24. We are now prepared to explain the method of multiplying by a number which is wholly expressed by digits.

Let it be required, for example, to multiply 789 by 453. The meaning here is, that the multiplicand 789 is to be taken 400 times *plus* 50 times *plus* 3 times. Now, it is obvious, that if 789 be mutiplied successively by 400, by 50, and by 4, and these partial products be added together, their sum will express the total product of the two given numbers. Hence the operation will stand as follows:

789
453

2367 the product of 789 by 3,
39450 - - - - - - 789 by 50,
315600 - - - - - - 789 by 400,

357417 equal to the *sum* of the partial products, or the total product of 789 by 453.

The first partial product is obtained as in Ex. 2 of No. 21, the second and third as in the Ex. No. 23.

Again, let it be required to find the product of 8795 by 5004.

Here the multiplicand 8795 is to be taken 5000 times *plus* 4 times.

8795
5004

35180 the product of 8795 by 4.
43975000 - - - - 8795 by 5000.

44010180 equal to the sum of the partial products, or the total product of 8795 by 5004.

From a consideration of the foregoing examples, it is manifest, that the ciphers, on the right of the partial products, cannot affect the sum of the latter, that is, the total product; these ciphers, therefore, which merely indicate the rank of the digits, may be omitted in practice, by observing, that the product of the units of the multiplicand by the tens of the multiplier, will be *tens*, and will therefore occupy the second place from the right; that the product of the units of the multiplicand, by the hundreds of the multiplier, will be *hundreds*, and will therefore occupy the third place from the right; or, in general, that the product of the units of the multiplicand, by any one of the digits of the multiplier, will express the same order of units as this digit, and will therefore stand exactly under it.

25. If the multiplier be a composite number, *that is*, one formed by the continual multiplication of two or more numbers, the product may be obtained by multiplying continually by its factors. Thus, for instance, 95 multiplied by 28, will be equal to 95 multiplied by 7, and this last product by 4. This must be obvious, because 4 times 7 times 95, are evidently 28 times 95.

26. The preceding method will apply, so far, although the multiplier is a *prime* number, *that is*, one which cannot be formed by the continual multiplication of any numbers. In this case, we have merely to multiply by the factors of that composite number which is nearest the given multiplier, and to or from this product, add or subtract that number of times the multiplicand, expressed by the units in the difference of the composite number and given multiplier, according as the former is less or greater than the latter. This will be better understood, perhaps, by an example. Suppose the product of 83 by 47 were required; then, since 47 is equal to 45 *plus* 2, or 48 *minus* 1, we might either multiply continually by the component factors of 45, and to this product add twice 83; or, multiply continually by the component factors of 48, and from this product subtract 83.

This latter method is applied with peculiar effect, when the multiplier is any number of nines. If the multiplier were 999 for example, we would merely require to annex three ciphers to the multiplicand, and then subtract it from the product; because 999 is equal to 1000 *minus* 1.

27. Multiplication may be proved in a variety of ways, many of which will readily occur to the intelligent student. One method is to make the factors change places, and see

whether their product, when taken in this order, be the same as at first.

This method of proof proceeds on the principle, that the product of two numbers is the same in whatever order they are taken; that 4 multiplied by 3, for example, will give the same product as 3 multiplied by 4. To demonstrate this, let the units in 4 be represented by a horizontal row of physical points, and repeat this row as often as there are units in 3. Then, it is obvious, that the whole number of points will be found, either by adding three *fours* together, or four *threes*. But three fours are equal to 4 multiplied by 3, and four threes are equal 3 multiplied by 4. Hence the product of 4 by 3, is equal to the product 3 by 4; and the same may be demonstrated of any two numbers whatever.

. . . .
. . . .
. . . .

Multiplication may be also proved by help of Nos. 25 and 26.

DIVISION.

28. Division is an expeditious method of finding how many times a less number can be subtracted from a greater; or, which amounts to the same thing, it is that process by which we discover how often the greater contains the less.

The less of the two given numbers is called the *divisor;* the greater, the *dividend;* and the number of times that the latter contains the former, is denominated the *quotient*

Since the quotient expresses the number of times that the dividend contains the divisor, it is manifest, that the product of the divisor and quotient must be equal to the dividend. Hence division has sometimes been defined, to be the method of finding a number, which, when multiplied by the less of two given numbers, shall produce the greater.

When the quotient, or number of times that the dividend contains the divisor is not very great, it may be obtained by actually performing the successive subtractions. Thus, let it be required to find how often 7 is contained in 35. Here by a process of continued subtraction, we find that 7 may be taken away 5 times from 35, and that after 5 abstractions there is no remainder. Hence we infer that 35 contains 7, 5 times.

$$\begin{array}{r} 35\\ 7\\ \hline 28\\ 7\\ \hline 21\\ 7\\ \hline 14\\ 7\\ \hline 7\\ 7\\ \hline \end{array}$$

29. This method of finding the quotient by continued subtraction, though possible in every instance, would evidently be often very tedious and inconvenient; we have therefore

recourse to another, by which the quotient may be determined in all cases, with great facility. And here it will be most convenient to conceive the quotient as that number, which, when multiplied by the divisor, will produce the dividend.

Ex. 1. Let it be required to find how often 2 is contained in 864.

$$\begin{array}{r} 2)864 \\ \hline 432 \end{array}$$

Here we readily perceive that 800 contains 2, 400 times, because 400 multiplied by 2 are equal to 800. In the same manner 60 contains 2, 30 times, and 4 contains 2, 2 times. Hence 2 is contained 400 *plus* 30 *plus* 2 times, or 432 times in 864.

30. A simpler view might perhaps be taken of division than the preceding; for it must appear obvious to every one, that the quotient of a number divided by 2 will be the half of that number; or, if divided by 3, the third part of that number, and so on; and it is as evident, that to obtain the half, the third, or fourth part, &c. of any number, we have merely to take the half, the third, or fourth part, &c. of every collection of its units. Hence, in the preceding example, we shall have for the quotient the half of 8 hundreds, or 8 units of the third degree, which are 4, *plus* the half of 6 tens, or 6 units of the second degree, which are 3, *plus* the half of 4 units of the first degree, which are 2.

Ex. 2. How often is 5 contained in 9675?

9 units of the fourth order, are equal to 5 *plus* 4, or 5 of the fourth order, and 40 of the third; these 40, together with the 6, make 46 of the third, which are equal to 45 of the third, and 10 of the second; these 10, together with the 7, are equal to 17 of the second, or 15 of the second and 20 of the first; which, together with the 5, make 25 of the first. Hence it appears, that the given number, 9675, is expressed by 5 units of the fourth order, 45 of the third, 15 of the second, and 25 of the first, thus:— And by taking the fifth part of each of these collections, we obtain for the quotient 1935.

5	45	15	25
1	9	3	5

This decomposition may be effected while the division is going on, by observing that the denomination of each remainder is ten times as great as that of the succeeding unit, thus:—

$$\begin{array}{r} 5)9675 \\ \hline 1935 \end{array}$$

The fifth part of 9 is 1, leaving a remainder of 4; this 4 being 40 of the succeeding units, will, with 6, make 46, the fifth part of which is 9, leaving a remainder of 1, which, in like manner with the 7, will make 17, the fifth part of which is 3, leaving a remainder of 2;

the 2 being 20 units of the succeeding order, will, with 5, make 25, the fifth part of which is 5.

This is the plan which is adopted in practice, when the divisor does not exceed 12. When the divisor is considerably great, it will be necessary that the whole numerical process appear, with the view of disburdening the memory, and insuring an accurate result.

Let it be required, to find, for example, how often 35 is contained in 25760.

$$\begin{array}{rl}
35)25760(& 7\text{ hundreds}\\
\underline{245} & \\
126(& 3\text{ tens}\\
\underline{105} & \\
210(& 6\text{ units}\\
\underline{210} &
\end{array}$$

From a comparison of the divisor with the highest order of units in the dividend, we see that there can be no unit either of the fifth or fourth order in the quotient. Our next inquiry then is, what is the 35th part of 257 units of the third order, *that is*, of 257 hundreds; this we perceive to be 7 hundreds, which, when multiplied by 35, and the product subtracted from 257, leaves a remainder of 12 hundreds, or 120 tens; these 120 tens, together with the 6 tens, make 126 tens, the 35th part of which is 3 tens, which, when multiplied in like manner by 35, and the product subtracted from 126 tens, leaves a remainder of 21 tens, or 210 units, the 35th part of which is exactly 6 units. Thus it appears, that the 35th part of the given number, 25760, is 7 hundreds, 3 tens, and 6 units, or 736; but the 35th part of the dividend (according to what is stated at the beginning of the present number) expresses the number of times that it contains the divisor; hence 25760 contains 35, 736 times.

Although the quotient in the preceding example has been obtained analytically, yet it is manifest that the mere mechanical process differs in nothing from that of the common rule the reason of which must here appear to every one.

This might also be made to appear in the following manner:—Suppose we wish to ascertain the quotient of 5052925 divided by 563.

$$\begin{array}{rrl}
563)5052925(& 8\cancel{0}\cancel{0}\cancel{0} & \text{the first partial quotient.}\\
\underline{4504\cancel{0}\cancel{0}\cancel{0}} & 9\cancel{0}\cancel{0} & \text{the second.}\\
5489\cancel{2}\cancel{5} & 7\cancel{0} & \text{the third.}\\
\underline{5067\cancel{0}\cancel{0}} & \underline{5} & \text{the fourth.}\\
4222\cancel{5} & 8975 & \text{the sum of the partial quotients, or the}\\
\underline{3941\cancel{0}} & & \text{whole quotient.}\\
2815 & & \\
\underline{2815} & &
\end{array}$$

If, in the above example, the ciphers be suppressed, which merely serve to indicate the rank of the digits, and one figure of the dividend be annexed to the remainder after each division, we arrive again at the common process, which is, as we now see, established on sound principle.

31. In Division, there are several abbreviations, &c. similar to those stated under Multiplication, (Nos. 22, 23, and 25,) which it would be superfluous here to exemplify.

32. Division may be proved, by observing whether the product of the divisor and quotient, added to the remainder, if there be any, be equal to the dividend.

FRACTIONS.

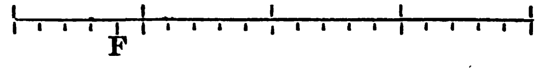
33. A fraction is a quantity which represents a part or parts of a unit or whole.

If a unit be divided into two equal parts, each of these is called a half of a unit or whole; if into three equal parts, each of them is called a third, &c.

34. A Fraction is expressed by two terms, called the *Numerator* and *Denominator*. The numerator shows of how many equal parts the fraction is composed, and the denominator indicates the number of these equal parts into which the unit is divided. Thus, for instance, if a unit be divided into *seven* equal parts, and *five* of these be taken, then these *five-sevenths* are expressed thus, $\frac{5}{7}$; the numerator being always placed above the denominator.

35. A fraction may be considered as the quotient arising from the division of the numerator by the denominator. Thus, for example, the value of the fraction $\frac{4}{5}$ is the same, whether we conceive it as denoting the fifth part of four units, or four-fifths of one unit. This may be easily shown by the help of a line.

Let each of the lines AB, BC, CD, and DE, represent a unit, then will AE be four units. Divide each of these units into five equal parts; and let AF be four of these, then is AF $\frac{4}{5}$ of AB, *that is*, AF is $\frac{4}{5}$ of a unit. Now, since



each of the five units is divided into five equal parts; the whole line AE is divided into twenty equal parts; AF, which contains four of these, is therefore the fifth part of AE, that is, of four times AB. Hence, it appears, that the fifth part of four times AB, is of the same value as four-fifths of once AB.

36. A *proper fraction* is one whose numerator is less than its denominator; as $\frac{5}{8}$.

37. An *improper fraction* is one whose numerator is either equal to, or greater than its denominator; as $\frac{5}{5}$ or $\frac{7}{4}$.

38. A *compound fraction* is a fraction of a fraction; as $\frac{2}{3}$ of $\frac{4}{5}$.

39. A *mixed number* is that which consists of a whole, or integer number, and a fraction; as $5\frac{3}{4}$.

From the preceding definitions, the following deductions are easily made.

40. (1.) That a fraction is equal to, less or greater, than unity, according as its numerator is equal to, less or greater, than its denominator.

41. (2.) That to multiply a fraction by any number, multiply the numerator by that number, and retain the same denominator; or divide the denominator, and retain the same numerator.

42. (3.) That to divide a fraction by any number, divide the numerator by that number, and retain the same denominator; or multiply the denominator, and retain the same numerator.

Hence it follows, that if the numerator and denominator of a fraction, be either both multiplied, or both divided by the same number, the value of the fraction will remain unchanged.

Thus $\frac{24}{36}$[*]$=\frac{12}{18}=\frac{6}{9}=\frac{2}{3}$; $\frac{2}{5}=\frac{4}{10}=\frac{6}{15}=\frac{8}{20}$, &c.

43. From the preceding, it is obvious, that the same fractional value may be expressed in an endless variety of ways. It hence becomes an important object to ascertain what the lowest or least terms are, in which it can be expressed. These may be attained, by dividing the terms of the fraction continually by some small number that will divide both, until they are not any farther divisible; or by dividing them at once by their greatest common measure, *that is*, by the greatest number that will divide both without a remainder.

44. The method of finding the greatest common measure of two numbers depends on the two following obvious prin-

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[*] For an explanation of the signs +, —, ×, ÷, =, see the beginning of Algebra.

ciples. 1. " Any number that measures each of two others, must also measure their sum and difference. 2. Any number that measures another, must also measure its double, treble, or any other multiplier of it."

Let it be required to find the greatest common measure of 325 and 455.

Here, since the number we are in search of must measure each of the numbers 325 and 455, it must, according to principle, 1*st*, measure their difference, 130; now, by principle, 2*d*, it must measure twice 130, or 260, which is the nearest multiple of 130 to 325. Again, since it measures both 325 and 260, it must measure their difference, 65. The greatest common measure then of the two given numbers must measure both 65 and 130. Now, 65 itself does this; hence 65 is the greatest common measure sought. It is obvious that it can be no number greater than 65, for no greater number would fulfil the requisite condition of dividing 65 without a remainder.

From the preceding investigation, it appears that the process of finding the greatest common measure, consists in the successive decomposition of the two proposed numbers, and may be exhibited as follows:

```
325)455(1
    325
    ---
    130)325(2
        260
        ---
         65)130(2
            130
            ---
```

Hence the general rule may be expressed thus: *Divide the greater by the less, then divide the preceding divisor by the remainder, and so on continually until* 0 *remains; the last divisor is the greatest common measure sought.*

Hence the greatest common measure of three or more numbers may be found. For we have merely to find the greatest common measure of the first two numbers, then the greatest common measure of *their* common measure and the third number, and so on to the last of them.

If, by the rule as now enunciated, we find the greatest common measure of the numerator and denominator of a fraction, then the fraction will be reduced to its lowest terms at once, by dividing both terms by it. Thus, if the terms of the fraction $\frac{325}{455}$ be both divided by 65, their greatest common measure, then this same fractional value will be expressed, in its lowest terms, by $\frac{5}{7}$. When, by the preceding process, we discover that the terms of a fraction have no common measure greater than unity, then the fraction is already in its lowest terms.

45. A whole number may be represented as a fraction having any given denominator.

Ex. Reduce 7 to a fraction whose denominator shall be 5. Here 7 are equal to 7 units, or $\frac{7}{1}$; this value will not be altered (No. 42,) if both numerator and denominator are multiplied by the same number; let them both be multiplied by 5, and we shall obtain $\frac{7\times 5}{1\times 5}$ or $\frac{35}{5}$ for the fraction sought. From this it is obvious that, to reduce a whole number to a fraction having a given denominator, we have merely to multiply the given number by the given denominator, and under their product write the denominator.

46. Hence it follows that a mixed number may be reduced to an improper fraction, and conversely.

Thus, $8\frac{7}{9} = \frac{8\times 9}{9}$ and $\frac{7}{9} = \frac{72}{9}$ and $\frac{7}{9}$ that is to $\frac{79}{9}$.

And $\frac{75}{8} = \frac{72}{8}$ and $\frac{3}{8} = 9\frac{3}{8}$.

A little attention to the preceding steps renders the formal statement of any rule quite unnecessary.

47. A compound fraction may be reduced to a simple one, by multiplying all the numerators together for the numerator of the simple fraction, and all the denominators together for its denominator.

The reason of this will appear from an example. Let it be required to reduce $\frac{2}{3}$ of $\frac{4}{5}$ to a simple fraction.

The meaning of this expression is, that the third part of $\frac{4}{5}$ is to be taken twice. Now, the third part of $\frac{4}{5}$ is equal to $\frac{4}{5} \div 3 = \frac{4}{5\times 3} = \frac{4}{15}$, (according to No. 42,) and $\frac{4}{15}$ taken twice is equal to $\frac{4}{15}\times 2 = \frac{8}{15}$, (No. 41,) a simple fraction, whose numerator is the product of the numerators, and denominator that of the denominators. If the compound fraction consist of three or more fractions, the two first may be reduced to a simple one, and then this and the third to another, and so on to the last.

Ex. 2. Reduce $\frac{1}{2}$ of $\frac{3}{5}$ of $\frac{5}{7}$ to a simple fraction.

Here $\frac{1}{2}$ of $\frac{3}{5}$ of $\frac{5}{7}=\frac{1\times3\times5}{2\times5\times7}=\frac{15}{70}=\frac{3}{14}$ by reducing it to its lowest terms.

An expression of this kind may often be simplified by expunging those numbers which are common to the numerator and denominator. Thus, in the preceding example, the number 5 may be cancelled, since it appears both in the numerator and denominator, for this is nothing else than the division of both terms by the same number.

Ex. 3. Reduce $\frac{3}{4}$ of $\frac{4}{7}$ of $\frac{7}{8}$ of $5\frac{2}{3}$ to a simple fraction.

The $5\frac{2}{3}=\frac{17}{3}$, (No. 46.) Hence $\frac{3}{4}$ of $\frac{4}{7}$ of $\frac{7}{8}$ of $5\frac{2}{3}=\frac{3}{4}$ of $\frac{4}{7}$ of $\frac{7}{8}$ of $\frac{17}{3}=\frac{\cancel{3}\times\cancel{4}\times\cancel{7}\times17}{\cancel{4}\times\cancel{7}\times8\times\cancel{3}}=\frac{17}{8}$.

48. Having given a fraction expressed in one denomination, it is sometimes convenient to express it in another. Thus, for example, let it be required to reduce $\frac{5}{7}$ of a guinea to the fraction of L. 1. We operate on this fraction exactly in the same way as we would on a whole number, to reduce it from guineas to pounds; *that is*, we multiply it by 21 to reduce it to shillings, and then divide by 20 to reduce the shillings to pounds, observing that the general rule for multiplying a fraction is to multiply its numerator; and for dividing it, to multiply its denominator. (Nos. 41 and 42.) Hence the operation will be as follows, $\frac{5}{7}$ of a guinea $=\frac{5\times21}{7\times20}=\frac{105}{140}=\frac{21}{28}=\frac{3}{4}$ of a pound.

Ex. 2. Reduce $\frac{5}{12}$ of half a guinea to the fraction of a crown.

Here the fraction must evidently be multiplied by 21, and divided by 10, because 21 sixpences makes half a guinea, and ten make a crown.

$\frac{5}{12}$ of half a guinea $=\frac{5\times21}{12\times10}=\frac{105}{120}=\frac{21}{24}=\frac{7}{8}$ of a crown.

49. A quantity may be reduced to a fraction of any given denomination belonging to it, by reducing the given quantity to the lowest name it contains for the numerator, and the proposed integer to the same name for the denominator, and then reducing the fraction to its lowest terms.

Ex. 1. Red. $2\frac{1}{2}$d. to the fraction of a shilling, $2\frac{1}{2}=5$ half-pence; and 1 sh. = 24 half-pence. Now, it is obvious, that 1 half-penny $=\frac{1}{24}$ of a shilling; and hence, that 5 half-pence, or $2\frac{1}{2}$d. $=\frac{5}{24}$ sh.

Ex. 2. Red. 17s. 6d. to the fraction of a £. 17s. 6d. = 35 sixpences; and £1 = 40 sixpences; hence 1 sixpence is $\frac{1}{40}$ of a £; and, of course, 35 sixpences are $\frac{35}{40}$ or $\frac{7}{8}$ of a £.

Ex. 3. Red. 3s. $4\frac{1}{2}$d. to the fraction of a guinea, 3s. $4\frac{1}{2}$d. = 81 half-pence; and 1 guinea = 504 half-pence; hence, 3s. $4\frac{1}{2}$d. $=\frac{81}{504}=\frac{27}{168}=\frac{9}{56}$ of a guinea.

50. The converse of this problem is also easily managed, viz. given a fraction to find its value. Thus, let it be required to find the value of $\frac{5}{8}$ of a guinea.

$\frac{5}{8}$ of a guinea is the same as the eight part of 5 guineas, (No. 35.) Now, 5 guineas are equal to 105 shillings; and hence, the eight part of 5 guineas is $=\frac{105}{8}=$ 13s. $1\frac{1}{2}$d.

51. In order that fractions may be compared or added together, it is necessary that they be of the same kind, *that is*, that they have the same denominator. Fractions of different denominators may be reduced to equivalent ones, having the same denominator, by multiplying each numerator into all the denominators except its own, for the numerators of the equivalent fractions, and all the denominators together for their common denominator.

The principle of this rule is, that if the numerator and denominator be both multiplied by the same number, the value of the fraction is not altered. That both terms are multiplied by the same number, by applying the preceding rule, will readily appear from an example.

Let the fractions $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$ be reduced to equivalent fractions, having the same denominator.

$$\frac{3}{4}=\frac{3\times6\times8}{4\times6\times8}=\frac{144}{192};\ \frac{5}{6}=\frac{5\times4\times8}{6\times4\times8}=\frac{160}{192};\ \frac{7}{8}=\frac{7\times6\times4}{8\times6\times4}=\frac{168}{192}.$$

Hence, $\frac{3}{4}, \frac{5}{6}$ and $\frac{7}{8}$ are $=\frac{144}{192}, \frac{160}{192}$ and $\frac{168}{192}$ respectively.

Fractions admit of being reduced to a common denominator, which is less than the product of their denominators, when the least common multiple of the denominators is less than their product.

Thus, in the example now given, it is obvious that the least common multiple of the denominators is 24, *that is*, 24 is the least number that will contain them all without a remainder. Now, 24 contains 4, 6 times; 6, 4 times, and 8, 3 times. Hence, if the terms of the fraction $\frac{3}{4}$ be multiplied 6, those of $\frac{5}{6}$ by 4, and those of $\frac{7}{8}$ by 3, we obtain $\frac{18}{24}$, $\frac{20}{24}$ and $\frac{21}{24}$ for the given fractions reduced to their least common denominator.

This method of reducing fractions to their least common denominator is often exceedingly convenient; the student would therefore do well to make himself thoroughly acquainted with it.

ADDITION OF FRACTIONS.

52. If the fractions to be added have the same denominator, their addition will be effected by writing the sum of their numerators above this denominator.

Ex. 1. $\frac{2}{11}+\frac{3}{11}+\frac{5}{11}=\frac{2+3+5}{11}=\frac{10}{11}$. This must appear obvious to every one, since 2, 3, and 5 quantities of the same kind, are equal to 10 of that kind. If the fractions have different denominators, they must be reduced to others having the same denominator before they can be added.

Ex. 2. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}=\frac{6}{12}+\frac{8}{12}+\frac{9}{12}$ (No. 51.) $=\frac{6+8+9}{12}=\frac{23}{12}=1\frac{11}{12}$ (No. 46.)

Ex. 3. $\frac{3}{4}$ of $\frac{5}{6}+\frac{2}{3}$ of $5\frac{1}{2}=\frac{3}{4}$ of $\frac{5}{6}+\frac{2}{3}$ of $\frac{11}{2}=\frac{15}{24}+\frac{22}{6}$ (No. 47.) $=\frac{15}{24}+\frac{88}{24}$ (No. 51.) $=\frac{15+88}{24}=\frac{103}{24}=4\frac{7}{24}$.

Ex. 4. $8\frac{5}{6}+7\frac{2}{3}+5\frac{7}{12}=8+7+5+\frac{5}{6}+\frac{2}{3}+\frac{7}{12}=20+\frac{10}{12}+\frac{8}{12}+\frac{7}{12}=20+\frac{25}{12}=20+2\frac{1}{12}=22\frac{1}{12}$.

Ex. 5. Required the sum of £ $\frac{5}{8}$, $\frac{8}{9}$ of a guinea, and $\frac{3}{4}$ of a shilling.

$$\left.\begin{array}{lrrr}
£\ \frac{5}{8}=\frac{5\times 20}{8}=\frac{100}{8}= & £\,0 & 12 & 6 \\
G.\ \frac{8}{9}=\frac{8\times 21}{9}=\frac{168}{9}= & 0 & 18 & 8 \\
S.\ \frac{3}{4}=\frac{3\times 12}{4}=\frac{36}{4}= & 0 & 0 & 9
\end{array}\right\}\ \text{(No. 50.)}$$

$$£\,1\quad 11\quad 11$$

The preceding example might be solved, by reducing the given fractions to others of the same kind, for instance to the fraction of a £, and then adding them all together thus: $\frac{8}{9}$ of a guinea $=\frac{8\times21}{9\times20}=\frac{168}{180}=\frac{42}{45}=\frac{14}{15}$ of a £, and $\frac{3}{4}$ of a shilling $=\frac{3}{4\times20}=\frac{3}{80}$ of a £ (No. 48.)

Hence $£\frac{5}{8}+\text{G.}\frac{8}{9}+\text{Sh.}\frac{3}{4}=£\frac{5}{8}+£\frac{14}{15}+£\frac{3}{80}=\frac{150}{240}+\frac{224}{240}+\frac{9}{240}=\frac{150+224+9}{240}=\frac{383}{240}=£1$, 11s. 11d. as before.

SUBTRACTION OF FRACTIONS.

53. If the fractions have the same denominator, their difference is equal to the difference of their numerators, written above this denominator.

Ex. 1. $\frac{7}{9}-\frac{2}{9}=\frac{5}{9}$, this is manifest; for two quantities of any kind subtracted from 7 of the same, will leave 5 of that kind.

If the fractions have different denominators, they must be reduced to other equivalent ones, having the same denominator, before the subtraction can take place.

Ex. 2. $\frac{4}{5}-\frac{3}{4}=\frac{16}{20}-\frac{15}{20}$ (No. 51) $=\frac{16-15}{20}=\frac{1}{20}$.

Ex. 3. $\frac{2}{3}$ of $5\frac{6}{7}-\frac{5}{6}$ of $2\frac{7}{8}=\frac{2}{3}$ of $\frac{41}{7}-\frac{5}{6}$ of $\frac{23}{8}=\frac{82}{21}-\frac{115}{48}$ (No. 47.) $=\frac{3936}{1008}-\frac{2415}{1008}=\frac{3936-2415}{1008}=\frac{1521}{1008}=\frac{507}{336}=\frac{169}{112}=1\frac{57}{112}$.

Ex. 4. Required the difference of $8\frac{7}{8}$ and $5\frac{11}{12}$.

$$\begin{array}{r} 8\frac{7}{8}=8\frac{21}{24} \\ 5\frac{11}{12}=5\frac{22}{24} \\ \hline 2\frac{23}{24} \end{array}$$

Having reduced the fractional parts of the quantities, to fractions of the same denominator, we find that the fractional part of the minuend exceeds that of the subtrahend. In order, therefore, to effect the subtraction, we add a unit, or $\frac{24}{24}$ to the $\frac{21}{24}$ of the minuend, which will make $\frac{45}{24}$; $\frac{22}{24}$ being subtracted from this, there remains $\frac{23}{24}$. Now, as the minuend was increased by unity, the subtrahend must also be increased by unity, that the difference may remain the same, (Principle 3, No. 18.) We shall therefore have 6 to subtract from 8, which leaves a remainder of 2. Hence the difference of the two given quantities is $2\frac{23}{24}$.

Ex. 5. From £$\frac{7}{12}$, take $\frac{3}{8}$ of a guinea.

$$\left.\begin{array}{l} £\tfrac{7}{12} = 11\text{s. } 8\text{d.} \\ \text{G.}\tfrac{3}{8} = 7\text{s. } 10\tfrac{1}{2}\text{d.} \end{array}\right\} \text{(No. 50.)}$$

$$\overline{\quad 3\text{s. } 9\text{d}\tfrac{1}{2}. \quad}$$

This example may be done in a way similar to that in which the last example of addition is solved.

$$\frac{3}{8}\text{ of a guinea} = \frac{3\times 21}{8\times 20} = \frac{63}{160} \text{ of a £ (No. 48.)}$$

$$\text{Hence } £\frac{7}{12} - \text{G.}\frac{3}{8} = £\frac{7}{12} - £\frac{63}{160} = \frac{280}{480} - \frac{189}{480} = \frac{280-189}{480} =$$

$$£\frac{91}{480} = 3\text{s. } 9\tfrac{1}{2}\text{d. as before.}$$

MULTIPLICATION OF FRACTIONS.

54. The rule for the multiplication of a fraction by an integer has been already stated, (No. 41.)

To multiply one fraction by another is to take such a part or parts of the one as the other expresses. Thus, for example, to multiply $\frac{5}{7}$ by $\frac{2}{3}$, means that the $\frac{1}{3}$ of $\frac{5}{7}$ is to be taken twice, or, which is the same thing, that $\frac{5}{7}$ is to be multiplied by 2 and divided by 3. Hence the process is the same as that of reducing a compound fraction to a simple one. The product of $\frac{5}{7}$ by $\frac{2}{3}$ is therefore $= \frac{2}{3}$ of $\frac{5}{7} = \frac{2\times 5}{3\times 7} = \frac{10}{21}$. (No. 47.)

Ex. 2. $6\frac{5}{8}\times 2\frac{2}{3} = \frac{53}{8}\times\frac{8}{3} = \frac{53}{3} = 17\frac{2}{3}$.

Ex. 3. $\frac{1}{2}$ of $\frac{2}{3}$ of $4\frac{1}{2}\times 6\frac{5}{8} = \frac{1}{2}$ of $\frac{2}{3}$ of $\frac{9}{2}\times\frac{53}{8} = \frac{1\times 2\times 9}{2\times 3\times 2}\times\frac{53}{8}$ $= \frac{9}{6}\times\frac{53}{8} = \frac{3}{2}\times\frac{53}{8} = \frac{159}{16} = 9\frac{15}{16}$.

Ex. 4. Required the price of $3\frac{3}{5}$ yards at 16s. 8d. per ell. Here we may find the price of a yard, and then multiply it by $3\frac{3}{8}$; or, we may reduce $3\frac{3}{8}$ yards to the fraction of an ell, and then multiply the 16s. 10d. by it.

Under the impression that it is more profitable for the student to understand one example thoroughly, than to run through a multitude carelessly, we shall solve the question by the two methods alluded to.

Since a yard contains 4 quarters, and an ell 5, a yard is, of course, $\frac{4}{5}$ of an ell; and, therefore, the price of a yard will be $\frac{4}{5}$ of 16s. 8d. = 13s. 4d. or $13\frac{1}{3}$. (No. 49.)

Hence $13\frac{1}{3}\times 3\frac{3}{8}=\frac{40}{3}\times\frac{27}{8}=\frac{1080}{24}=45$s.$=$ £2, 5s.

Otherwise, $3\frac{3}{8}$ yards $=\frac{27}{8}$ yards $=\frac{27\times 4}{8\times 5}=\frac{108}{40}=\frac{27}{10}$ ells. (No. 48.)

Hence 16s. 8d. $\times\frac{27}{10}=16\frac{2}{3}\times\frac{27}{10}=\frac{50}{3}\times\frac{27}{10}=\frac{1350}{30}=45$s.$=$ £2, 5s. as before.

DIVISION OF FRACTIONS.

55. The manner of dividing a fraction by an integer has been already explained, (No. 42.)

When the divisor is a fraction, the rule is to invert the terms of the divisor, and then multiply the dividend by this inverted divisor, as in Multiplication.

The reason of this rule will be perceived by the following examples.

Let it be required to divide $\frac{7}{8}$ by $\frac{2}{3}$.

$\frac{7}{8}\div 2=\frac{7}{16}$, (No. 42.); now, as the divisor is not 2, but $\frac{2}{3}$, or $\frac{1}{3}$ of 2, (No. 35.), $\frac{2}{3}$ will therefore be contained three times as often in $\frac{7}{8}$ as 2 is; hence $\frac{7}{16}\times 3=\frac{21}{16}=1\frac{5}{16}$ is the quotient, which agrees with the rule.

The quotient may be obtained in another way, though less convenient than the preceding.

Thus, $\frac{7}{8}\div\frac{2}{3}=\frac{21}{24}\div\frac{16}{24}$. Now, it is obvious that $\frac{21}{24}$ will contain $\frac{16}{24}$ just as often as 21 contains 16, which is $\frac{21}{16}$ or $1\frac{5}{16}$ times.

Ex. 2. $7\frac{5}{6} \div 3\frac{2}{3} = \frac{47}{6} \div \frac{11}{3} = \frac{47}{6} \times \frac{3}{11} = \frac{141}{66} = \frac{47}{22} = 2\frac{5}{22}$.

Ex. 3. $\frac{6}{7}$ of $5\frac{3}{5} \div 2\frac{1}{2} = \frac{6}{7}$ of $\frac{28}{5} \div \frac{5}{2} = \frac{168}{35} \div \frac{5}{2} = \frac{24}{5} \div \frac{5}{2} = \frac{24}{5} \times \frac{2}{5} = \frac{48}{25} = 1\frac{23}{25}$.

Ex. 4. Divide £$\frac{5}{6}$ by $3\frac{1}{2}$. £$\frac{5}{6} \div 3\frac{1}{2} = \frac{5}{6} \div \frac{7}{2} = \frac{5}{6} \times \frac{2}{7} = \frac{10}{42}$£ $= \frac{5}{21}$£ $=$ 4s. $9\frac{1}{4}$d.

In division of fractions there are sometimes results which wear the air of a paradox; but when the student reflects that the quotient expresses the number of times that the dividend contains the divisor, these difficulties will disappear. $\frac{1}{2} \div \frac{1}{8}$, for example, should evidently give 4 as a quotient, since $\frac{1}{2}$ is $= \frac{4}{8}$, and will therefore contain $\frac{1}{8}$, 4 times. By the rule $\frac{1}{2} \div \frac{1}{8} = \frac{1}{2} \times \frac{8}{1} = \frac{8}{2} = 4$.

$\frac{4}{5} \div \frac{1}{3} = \frac{4}{5} \times \frac{3}{1} = \frac{12}{5} = 2\frac{2}{5}$, *meaning*, that $\frac{4}{5}$ contains $\frac{1}{3}$, $2\frac{2}{5}$ times.

$6 \div \frac{2}{5} = 6 \times \frac{5}{2} = \frac{30}{2} = 15$, *that is*, 6 contains $\frac{2}{5}$, 15 times.

EXERCISES IN FRACTIONS.

1. Reduce $\frac{525}{945}$ to its lowest terms. Ans. $\frac{5}{9}$.

2. Reduce 9 to a fraction whose denominator shall be 5. Ans. $\frac{45}{5}$.

3. Reduce $8\frac{7}{11}$ to an improper fraction. Ans. $\frac{95}{11}$.

4. Reduce $\frac{43}{9}$ to a mixed number. Ans. $4\frac{7}{9}$.

5. Reduce $\frac{2}{3}$ of $\frac{5}{6}$ of $\frac{3}{5}$ to a simple fraction. Ans. $\frac{1}{3}$.

6. Reduce $\frac{8}{9}$ of a guinea to the fraction of a £. Ans. $\frac{14}{15}$.

7. Reduce 12s. 6d. to the fraction of a £. Ans. $\frac{5}{8}$.

8. Required the value of $\frac{5}{12}$ of a guinea. Ans. 8s. 9d.

9. Reduce $\frac{3}{4}$, $\frac{5}{7}$, and $\frac{8}{9}$ to equivalent fractions having a common denominator. Ans. $\frac{189}{252}$, $\frac{180}{252}$, $\frac{224}{252}$.

10. Reduce $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{7}{8}$ to equivalent fractions having the least common denominator. Ans. $\frac{16}{24}$, $\frac{18}{24}$, $\frac{21}{24}$.

11. Add together the fractions $\frac{1}{2}$, $\frac{5}{6}$, and $\frac{11}{12}$. Ans. $2\frac{1}{4}$,

12. Add $\frac{7}{12}$ of a pound, $\frac{5}{6}$ of a guinea, and $\frac{5}{8}$ of a shilling together. Ans. £1, 9s. 9$\frac{1}{2}$d.

13. Add $\frac{2}{3}$ of 7$\frac{5}{6}$ and $\frac{3}{4}$ of 8$\frac{2}{3}$ together. Ans. 11$\frac{13}{18}$.

14. From $\frac{5}{6}$ subtract $\frac{1}{2}$. Ans, $\frac{1}{3}$.

15. From $\frac{3}{4}$ of $\frac{8}{9}$ subtract $\frac{4}{5}$ of $\frac{1}{2}$. Ans. $\frac{4}{15}$.

16. From $\frac{5}{6}$ of half a guinea subtract $\frac{8}{9}$ of a shilling.
Ans. 7s. 10$\frac{1}{4}$d. $\frac{1}{3}$

17. Multiply $\frac{7}{10}$ by 5. Ans. 3$\frac{1}{2}$.

18. Multiply $\frac{3}{5}$ of $\frac{10}{11}$ by $\frac{2}{3}$ of 6$\frac{1}{2}$. Ans. 2$\frac{4}{11}$.

19. Multiply 9$\frac{4}{5}$ by 7$\frac{5}{6}$. Ans. 76$\frac{23}{30}$.

20. Divide $\frac{3}{4}$ by $\frac{2}{3}$. Ans. 1$\frac{1}{8}$.

21. Divide 12 by $\frac{2}{3}$ of 6$\frac{3}{4}$. Ans. 2$\frac{2}{3}$.

22. Divide £$\frac{5}{6}$ by 5$\frac{2}{3}$. Ans. 2s. 11$\frac{1}{4}$d. $\frac{3}{17}$.

23. From a piece of cloth, a merchant cut off $\frac{1}{4}$, and then $\frac{2}{5}$ of the whole; after which there remained 35 yards. Required the length of the original piece. Answ. 100 yds.

24. A can do a piece of work in 8 days, and B in 10 days. In what time will A and B, working together, do it?
Ans. 4$\frac{4}{9}$ days.

25. A, B, and C can do a piece of work in 4 days. A by himself, can do it in 12 days, and B in 15; in what time will C do it? Ans. 10 days.

26. After paying away $\frac{1}{4}$ of my money, and then $\frac{1}{5}$ of the remainder, I had 18 guineas left; how much money had I at first? Ans. 30 guineas.

27. A, B, and C have a joint stock concern, of which $\frac{3}{8}$ belong to A, $\frac{1}{3}$ to B, and the remaining part to C. C having occasion to sell out, sold $\frac{4}{7}$ of his share to A, and the remainder to B. What fraction of the whole will now belong to A and B respectively? Ans. A $\frac{13}{24}$, and B $\frac{11}{24}$.

CONTINUED FRACTIONS.

56. The object of *Continued Fractions* is to find such approximate values of an irreducible fraction, as shall be expressed in less terms.

The nature and properties of these fractions will be most clearly unfolded by an example. Suppose that we have the fraction $\frac{216}{887}$, which, though it consists of several figures, does not admit of being reduced to lower terms. The value of this fraction will not be altered, if both numerator and denominator are divided by the same quantity. Let the divisor be the numerator 216. The result of the division will be $\frac{1}{4\frac{23}{216}}$. The value of the fraction then, it is obvious, is somewhere intermediate between $\frac{1}{4}$ and $\frac{1}{5}$; and if we neglect the small fraction $\frac{23}{216}$, $\frac{1}{4}$ may be considered as the *first approximation* to the value of the fraction $\frac{216}{887}$. The fraction $\frac{1}{4}$ is, however, too great. For the denominator is not 4, but 4 increased by an additional quantity; and consequently the true value is less than $\frac{1}{4}$.

To obtain a nearer approximation, we may divide both numerator and denominator of the fraction $\frac{23}{216}$ by 23, and the result is $\frac{1}{9\frac{9}{23}}$. So that the fraction $\frac{216}{887}$ may be represented by the fractions $\cfrac{1}{4\,\cfrac{1}{9\frac{9}{23}}}$. Neglecting the last fraction, $\frac{9}{23}$, $\frac{216}{887}$ will be equal to $\frac{1}{4\frac{1}{9}}$ nearly. $4\frac{1}{9}$ is equal to $\frac{37}{9}$, and 1 divided by $\frac{37}{9}$, is equal to 1 multiplied by $\frac{9}{37}$, or to $\frac{9}{37}$, which may be regarded as a *second approximation* to the value of the fraction $\frac{216}{887}$. And if the steps of the process be attentively considered, it will be easy to perceive that the approximation $\frac{9}{37}$ is too small. For the denominator 9 being employed instead of $9\frac{9}{23}$, the fraction $\frac{1}{9}$ added to the denominator 4 is too great, and consequently the last fraction is too small.

The continued fraction $\cfrac{1}{4\,\cfrac{1}{9\frac{9}{23}}}$ admits of being extended still farther, if we divide the last fraction by its numerator 9. The result will be $\cfrac{1}{4\,\cfrac{1}{9\,\cfrac{1}{2\frac{5}{9}}}}$, which, omitting the $\frac{5}{9}$, may be regarded as a *third approximation* to the value of the fraction $\frac{216}{887}$. To bring back the expression $\cfrac{1}{4\,\cfrac{1}{9\frac{1}{2}}}$ to a simple fraction, we proceed thus: $9\frac{1}{2}$ is equal to $\frac{19}{2}$, and $\frac{1}{9\frac{1}{2}}$ is equal to 1 divided by $\frac{19}{2}$, or equal to $\frac{2}{19}$. $4\frac{2}{19}$ is equal to $\frac{78}{19}$, and 1 divided by $\frac{78}{19}$ is equal to $\frac{19}{78}$. So that $\frac{19}{78}$ is the *third approximation* to the value of $\frac{216}{887}$. $\frac{19}{78}$ is, however, too great; for the fraction $\frac{1}{2}$ being employed instead of $\frac{1}{2\frac{5}{9}}$, forms when joined to 9 a denominator too great; and the fraction added to 4 will consequently be too small, and therefore the last fraction will be too great.

By continually dividing the last fraction by its numerator, the continued fraction may be extended still farther, until a fraction is found having unity for its numerator. It is unnecessary to trace each step minutely. An accurate idea of the whole process may be obtained from the mere inspection of the following statement, exhibiting the successive forms in which the fraction $\frac{216}{887}$ may be represented.

$$\begin{array}{ccccc}
(1.) & (2.) & (3.) & (4.) & (5.) \\
\cfrac{1}{4\frac{23}{216}} &
\cfrac{1}{4\;\cfrac{1}{9\frac{9}{23}}} &
\cfrac{1}{4\;\cfrac{1}{9\;\cfrac{1}{2\frac{5}{9}}}} &
\cfrac{1}{4\;\cfrac{1}{9\;\cfrac{1}{2\;\cfrac{1}{1\frac{4}{5}}}}} &
\cfrac{1}{4\;\cfrac{1}{9\;\cfrac{1}{2\;\cfrac{1}{1\;\cfrac{1}{1\frac{1}{4}}}}}}
\end{array}$$

Or, as it is sometimes written, adopting the symbols of algebra.

$$\cfrac{1}{4+\cfrac{1}{9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4}}}}}}$$

From what has been said, it will be easy to convert each of these continued fractions into a simple fraction, and thus to obtain an approximation to the true value.

The fraction originally was $\frac{216}{887}$.

The 1st approximation is $\frac{1}{4}$, which is too great by $\frac{1}{154}$.

The 2d ditto is $\frac{9}{37}$, which is too small by $\frac{1}{1591}$.

The 3d ditto is $\frac{19}{78}$, which is too great by $\frac{1}{13837}$.

The 4th ditto is $\frac{28}{115}$, which is too small by $\frac{1}{25501}$.

The 5th ditto is $\frac{47}{193}$, which is too great by $\frac{1}{170191}$.

The 6th ditto is $\frac{216}{887}$, which is the true value.

From this example, it would appear that the fractions derived from successive combinations of the members of a continued fraction, approximate gradually to the true value of that quantity from which the continued fraction has been derived. These fractions, it appears, also, are alternately greater and less than the true value, and therefore the value of the quantity will always be found between any two consecutive fractions.

57. By means of continued fractions, we are enabled to solve the following general problem. *A fraction expressed by a great number of figures being given, to find all the fractions, in less terms, which approach so near the truth, that it is impossible to approach nearer without employing greater ones.* A few examples will illustrate this.

Ex. 1. It is ascertained by astronomers, that the length of the *tropical* year is 365 days, 5 hours, 48 minutes, and 48 seconds, which is longer than the common year of 365 days, by 5 hours, 48 minutes, and 48 seconds, or 20,988 seconds. Now, one day is equal to 86,400 seconds; therefore the year exceeds 365 days by the fraction $\frac{20928}{86400}$; so that, at the end of 86,400 years, it is necessary to intercalate 20,928 days. It is required, therefore, to find a series of fractions, expressed in lower terms, which approximate to the fraction $\frac{20928}{86400}$.

To reduce $\frac{20928}{86400}$ to a continued fraction, we have to perform an operation precisely similar to that stated, (No. 44.)

```
20928)86400(4
      83712
      ─────
       2688)20928(7
            18816
            ─────
             2112)2688(1
                  2112
                  ────
                   576)2112(3
                       1728
                       ────
                        384)576(1
                            384
                            ───
                            192)384(2
                                384
                                ───
```

Hence we obtain the continued fraction—

$$\frac{1}{4+\frac{1}{7+\frac{1}{1+\frac{1}{3+\frac{1}{1+\frac{1}{2}}}}}}$$

And by combining the successive members into a simple fraction, we will obtain a series of fractions, each approaching more nearly than its predecessor to the true value. These are $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{31}{128}$, $\frac{39}{161}$, $\frac{109}{450}$.

The first fraction $\frac{1}{4}$ indicates that we must intercalate one day every four years, in order to preserve the correspondence between the common year and the year of the seasons. This intercalation was first made by Julius Cæsar: every fourth year has therefore 366 days, and is called Leap year. As the fraction $\frac{1}{4}$ is greater than the true value of the fraction $\frac{20928}{86400}$, the intercalation of one day in four years, renders the common year a little longer than the tropical year. A small error was thus introduced into the calendar. By continual accumulation, this error, in the year 1582, amounted to ten days, and thereby deranged the time of celebrating the church festivals. This error was rectified by Pope Gregory XIII. By his decree, he ordered the ten days, which had progressively accumulated from the time of Julius Cæsar, to be cut off from the year 1582; and, to prevent future accumulations of this kind, directed, that the last year of every century not divisible by 4, should be reckoned only a common year. So that the years 1700, 1800, 1900, &c. which, according to the Julian calendar, are leap years, are considered as common years in the Gregorian calendar.

The mode of reckoning by the Gregorian calendar, or *new style*, was not introduced into Great Britain until the year 1752. It is now adopted by all the European states except Russia, where the *old style*, or Julian mode of reckoning, still prevails. The difference between the two stiles at present amounts to twelve days, and will continually increase.

In the Gregorian calendar only 97 days are intercalated in 400 years. From the series of approximate fractions stated above, it is obvious that it would be more accurate to intercalate 109 days in 450 years. The error, however, is so slight, that it will not amount to a day during the lapse of many centuries.

The third approximate fraction mentioned above is $\frac{8}{33}$, intimating that 8 days must be intercalated in 33 years. It is a curious fact, that this was the correction proposed by Omar, an astronomer at the court of Persia, in the year 1079. This simple method differs from the truth only one minute in a century.

Ex. 2. It is ascertained that the diameter of a circle is very nearly $\frac{100000}{314159}$ of its circumference. It is required to find a series of fractions expressed in lower terms that shall approximate to this value.

Reducing $\frac{100000}{314159}$ to the form of a continued fraction, (Nos. 44 and 56,) we obtain,

$$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{25+\cfrac{1}{1+\cfrac{1}{7+\cfrac{1}{4}}}}}}}}$$

The 1st approximation is - - $\frac{1}{3}$

The 2d - $\cfrac{1}{3+\cfrac{1}{7}}$ - - - $\frac{7}{22}$

The 3d - $\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{15}}}$ - - $\frac{106}{333}$

The 4th - $\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1}}}}$ - - $\frac{113}{355}$

The 5th - $\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{25}}}}}$ - $\frac{2931}{9208}$, &c.

If great accuracy be not required, in any practical question connected with the circle, it may be sufficient to consider the diameter as $\frac{7}{22}$ of the circumference. But should greater exactness be requisite, then the fraction $\frac{106}{333}$, or still better $\frac{113}{355}$, will yield a more accurate result.

From the statement and examples now given, the manner of forming continued fractions, and likewise of obtaining a series of fractions expressed in lower terms, which continually approximate to the true value of the original fraction, will be easily apprehended.

To facilitate the formation of these approximate fractions, the following general rule may be given:

The first fraction will have unity for its numerator, and for its denominator the first quotient obtained by dividing, as in (No. 44.)

The second fraction will have for its numerator the second quotient, and for its denominator the product of the two quotients increased by unity.

And each successive approximation will be found by multiplying both terms of the last approximate fraction by the next quotient, and increasing these products respectively by the terms of the second last fraction.

We may often approximate to the value of an irreducible fraction, by making such a small change on its terms as shall render it reducible.

Thus $\frac{57}{113}$ is nearly equal to $\frac{56}{112}$, or $\frac{1}{2}$.

$\frac{271}{636}$ $\frac{273}{637} = \frac{3}{7}$.

EXERCISES ON CONTINUED FRACTIONS.

1. Required the successive approximations to the fraction $\frac{43}{95}$. Ans. $\frac{1}{2}$, $\frac{4}{9}$, $\frac{5}{11}$, $\frac{19}{42}$.

2. Required the successive approximations to the fraction $\frac{473}{986}$. Ans. $\frac{1}{2}$, $\frac{11}{23}$, $\frac{12}{25}$, $\frac{59}{123}$, $\frac{71}{148}$.

DECIMAL FRACTIONS.

57. Decimal fractions are those which have for their denominator 10, or 100, or 1000, or in general some power of 10.

The denominator of a decimal fraction is not generally expressed, but is indicated by a point prefixed to the numerator.

Thus $\frac{5}{10}$, $\frac{57}{100}$, $\frac{345}{1000}$, are respectively written .5, .57, .345; or sometimes 0.5, 0.57, 0.345, when the cipher shows that there is no integral digit.

Since $\frac{57}{100}$ are equal to $\frac{5}{10}+\frac{7}{100}$ and $\frac{345}{1000}$ to $\frac{3}{10}+\frac{4}{100}+\frac{5}{1000}$, &c. it is obvious that the figure next the decimal point will express *tenths*, the second *hundredths*, the third *thousandths*, and so on, diminishing in a tenfold proportion from left to right, the same as integer numbers.

58. From this remark, it will be easy to point the numerator, so that it shall express any decimal fraction whatever. For, suppose we have the fraction $\frac{5}{100}$, then as it is the second place from the decimal point which expresses *hundreths*, we prefix a cipher to the 5, in order that it shall occupy that place thus .05. In the same manner $\frac{5}{1000}$ will be written thus .005, because it is the third figure from the decimal point which expresses *thousandths*; $\frac{57}{100000}$ are equal to $\frac{5}{10000}+\frac{7}{100000}$, and will, therefore, be written thus, .00057, the fourth place from the decimal point being that of *ten thousandths*, and the fifth that of *hundred thousandths*. From these examples it is easy to perceive, that in every instance where the value of a decimal fraction is to be expressed by means of its numerator, that numerator must contain as many figures as there are ciphers in the denominator. If the numerator of itself does not contain so many figures, then the deficiency must be made up by prefixing ciphers to it. Thus in the fraction $\frac{57}{100000}$ there are five ciphers in the denominator and only two figures in the numerator, and hence three ciphers must be prefixed, that the digits may occupy their proper rank.

59. The converse of the problem enunciated in the preceding number will also be easily managed, *viz*. given a decimal fraction, expressed in the entire form, to express it in the ordinary fractional form. From what is there stated, it is

obvious that the denominator of a decimal fraction will be unity, with as many ciphers annexed as there are figures in the given numerator.

Thus .5, .043, .00307, will be changed into $\frac{5}{10}$, $\frac{43}{100}$, and $\frac{307}{100000}$ respectively.

60. Since the value of any figure of a decimal fraction depends entirely on the rank which it holds in regard to the decimal point; any number of ciphers may either be effaced or written on the right of a decimal without affecting its value. Thus, .5, for example, expresses the same value as .50 or .500; for .5, .50, .500, are respectively $\frac{5}{10}$, $\frac{50}{100}$, and $\frac{500}{1000}$, (No. 59,) each of which is equal to ½ according to No. 42. The annexing of ciphers on the right of a decimal, corresponds to the multiplication of the terms of the fraction by the same number, and the suppression of ciphers, to the division of the terms by the same number, in either of which cases the value of the fraction is not changed.

61. From what has been stated, it appears that decimal fractions are written exactly in the same way as integer numbers. Thus, for example, the number five hundred and forty-nine *units*, or which is the same thing, 5 *hundreds*, 4 *tens*, and 9 *units*, is written 549. In like manner, five hundred and forty-nine *thousandths*, or which is the same thing, 5 *tenths*, 4 *hundredths*, and 9 *thousandths*, is written .549, or with the cipher, 0.549.

An expression, in which there are both integers and decimals, is read by first enunciating the integral part, and then the fractional. Thus, 73.685, is read 73 *units* and 685 *thousandths*, or, simply, for the sake of conciseness, 73 decimal 685.

62. The great utility of decimal fractions, consists in their enabling us to express any numerical quantity, however small, on the decimal scale, and thus to subject fractional quantities to the same operations as integer numbers. It is in Logarithmic calculation, however, that their convenience is particularly felt.

63. The addition of decimals is performed exactly in the same way as that of integer numbers. The fractions to be added, are written in such a manner, that quantities of the same kind may stand under one another, and then their sum is obtained as in Simple Addition.

Ex. 1. Required the sum of 0.67, 0.087, and 0.0054.

$$\begin{array}{l} 0.67 \\ 0.087 \\ 0.0054 \\ \hline 0.7624 \text{ The sum.} \end{array}$$

Ex. 2. Required the sum of 4.86, 28.075, 52.904, and 0.0083.

$$\begin{array}{r@{.}l} 4&86 \\ 28&075 \\ 52&904 \\ 0&0083 \\ \hline 85&8473 \text{ The sum.} \end{array}$$

It is hardly necessary to add one remark in illustration of the preceding examples. In the last, the sum of the *thousandths* is 17, which are equal to 1 *hundredth* and 7 *thousandths.* The 1 hundredth being carried to its proper column, the sum will there be 14, which, in like manner, are equal to 1 *tenth* and 4 *hundredths.* The sum of the *tenths,* with the addition of the 1 carried, will be 18; now, 18 *tenths* are obviously equal to 1 *unit* and 8 *tenths,* the unit is of course carried to its proper column, &c. With regard to the place of the decimal point in the sum, it is manifest that it will be exactly under those of the several quantities to be added.

64. The method of performing subtraction in decimals, is also precisely the same as that in integer numbers.

Ex. 1. Required the difference of 0.76 and 0.4579. The value of the first number will not be altered if two ciphers be annexed to it, (No. 60.). Let this be done, then the operation will be as follows:—

$$\begin{array}{l} 0.7600 \\ 0.4579 \\ \hline 0.3021 \text{ The difference.} \end{array}$$

The object of annexing ciphers to the minuend, is to reduce it to a fraction, having the same denominator as the subtrahend. Thus, $0.76 = \frac{76}{100}$ and $0.4579 = \frac{4579}{10000}$ (No. 59.) Now, if the terms of the fraction $\frac{76}{100}$ be both multiplied by 100, (by which its value will not be altered, No. 42,) we get $\frac{7600}{10000}$, and hence $\frac{7600}{10000} - \frac{4579}{10000} = \frac{3021}{10000} = 0.3021$.

The ciphers are generally omitted in practice.

Ex. 2. From 47.042, subtract 9.70684.

```
47.042
 9.70684
────────
37.33516 The difference.
```

The same remark may be made here as in Addition, with respect to the place of the decimal point.

65. Since the decimal point separates the integral units from the decimal parts, its change of place will necessarily be accompanied by a change of value of the total number. In moving the decimal point towards the right, the number of integer units will be increased, and consequently the total value of the proposed number. In moving it towards the left, the number of integer units will be diminished, and hence also the value of the proposed number.

66. If the decimal point be moved one place toward the right, then the value of every figure will be increased 10 times, and consequently the value of the whole number will now be 10 times as great as at first. If it be moved two places toward the right, then, since by one removal, the value of the whole number is increased 10 times, another removal will increase this last value also ten times, and hence, by two removals, the original value will be increased 100 times. In like manner, by moving the decimal point three places to the right, the original value will be increased 1000 times, and so on.

From this it follows, that the multiplication of a number containing decimals, by 10, 100, 1000, or, in general, any power of 10, will be effected by removing the decimal point as many places toward the *right*, as there are ciphers in the given multiplier.

Thus 54.8674, when multiplied by 10, 100, 1000 respectively, will become 548.674, 5486.74, and 54867.4. In the same manner, 0.00784 when multiplied by 10, 100, 1000, will become 0.0784, 0.784, and 7.84 respectively.

67. Conversely, the division of any number by any power of 10, will be effected by removing the decimal point as many places toward the *left* as the given divisor contains ciphers.

Thus 8764.7, when divided by 10,100 and 1000 respectively, will become 876.47, 87.647, and 8.7647. In the same manner, 0.845, when divided by 10,100 and 1000, will become 0.0845, 0.00845, and 0.000845 respectively.

68. The preceding remarks will enable us to explain the rule given for pointing the product of two numbers containing decimals. The rule, as generally stated, is as follows:—Multiply the two numbers together as in multiplication of integer numbers, and from the product point off as many decimals from the right as are contained in both factors. (That is, if there be 3 decimals in the one factor, for example, and 2 in the other, then 3+2, or 5 decimals must be pointed off from the right; if there be 4 in the one, and 5 in the other, then 9 must be pointed off, &c.)

The reason of the rule as now enunciated will be readily perceived by attending to the following example:—Let it be required to multiply 76.45 by 9.349.

$76.45 = 76\frac{45}{100}$ (No. 59) $= \frac{7645}{100}$ (No. 46.) In the same manner $9.349 = 9\frac{349}{1000} = \frac{9349}{1000}$. Now, $\frac{7645}{100} \times \frac{9349}{1000} = \frac{71473105}{100000}$ (No. 54) $= 714.73105$ (No. 67;)—a result in which the number of decimal figures agree with that determined by the rule. The same way of reasoning will obviously lead to the same conclusion in every instance.

If one of the factors be an integer number, then there will be as many decimals in the product as are contained in the other factor.

Ex. 2. Required the product of 0.087 by 0.00476.

$0.087 = \frac{87}{1000}$, and $0.00476 = \frac{476}{100000}$, and $\frac{87}{1000} \times \frac{476}{100000} = \frac{41412}{100000000} = .00041412$ (No. 58.)

In such an example as the preceding the ciphers may be left out for the sake of convenience, until the product of the digits be obtained, after which as many must be prefixed as shall make up the proper number of decimals. Thus $87 \times 476 = 41412$. Now, as there are *three* decimal places in the one factor, and *five* in the other, there must be in all *eight* in the product, and therefore three ciphers must be prefixed to 41412, thus .00041412 for the true product.

69. The rule for pointing the quotient may be readily inferred from that for the product, as stated in the preceding number. For, since by the nature of division (No. 28,) the product of the quotient and divisor is equal to the dividend, it follows that there must be as many places of decimals in the dividend, as there are in the divisor and quotient toge-

ther; and hence the quotient must contain as many decimals, as the number of those in the dividend exceed those in the divisor.

Ex. 1. Divide 15248.91874 by 39.47.

```
39.47)15248.91874(386.342
      11841
      -----
       34079
       31576
       -----
        25031
        23682
        -----
         13498
         11841
         -----
          16577
          15788
          -----
            7894
            7894
            ----
```

After having obtained the quotient as in Simple Division, we observe that there are *five* places of decimals in the dividend, *two* in the divisor, and hence that there must be *three* in the quotient, which we accordingly point off from the right.

Ex. 2. Divide 0.00063504 by 0.084.

Here the ciphers may be omitted until the quotient of the digital parts be obtained, as in Ex. 2. of Multiplication, 63504 when divided by 84 yields a quotient of 756. Now, we see from comparing the number of decimal places in the dividend with those in the divisor, that there are *five* more in the former than in the latter, and hence that there must be *five* in the quotient. Two ciphers must, therefore, be prefixed to 756 thus, .00756, or, which is the same thing, 0.00756, in order to make up the proper number.

70. When the dividend does not contain the divisor an exact number of times, or, in other words, when the quotient cannot be expressed entirely in integer numbers, still it may either be expressed exactly, or at least with sufficient approximation, by means of decimals.

Let it be required to divide 4213 by 88.

After obtaining the partial quotient 47 in the ordinary way, there remains 77; then multiply this 77 by 10, (that is, annex a cipher to it,) in order to reduce it to *tenths*; the 770 tenths being divided by 88, yield a quotient of 8 tenths, with a remainder of 66. These 66 tenths are in like manner multiplied by 10, to reduce them to *hundredths*; the 660 hundredths, divided by 88, give a quotient of 7 hundredths, with a remainder of 44, to which another cipher is annexed to reduce them to thousandths; the 440 thousandths divided by 88 give 5 *thousandths.*

88)4213(47.875
352
693
616
770 *tenths*
704
660 *hundredths*
616
440 *thousandths*
440

In the preceding example the true quotient is 47.875. Although there are many instances in which the quotient cannot be expressed exactly, yet we can approximate to it within any assigned limits, and hence insure an accuracy sufficient for all practical purposes. The necessary degree of approximation will, of course, depend upon the magnitude of the unit in question, and the accuracy required. The first decimal place, after the integral part, if there be any, will give the quotient true to the tenth of the unit; the second place, to the hundredth of the unit; the third, to the thousandth of the unit, and so on.

71. A vulgar fraction may be easily reduced to a decimal; for we have merely to reduce the numerator to decimal parts, and then to divide by the denominator, as in the following example:—

Let it be required to express $\frac{7}{8}$ decimally.

This, according to our second mode of conceiving the meaning of a fraction, is 7 units divided by 8. Hence the operation will stand as follows:

A cipher is annexed to 7, to reduce it to tenths, which, being divided, leaves a remainder of 6, to which another cipher is annexed, to reduce the tenths to hundredths, and so on, either until the fraction is expressed in decimals, or at least until the remainder consists of parts so small, as that it may be totally neglected, without sensibly affecting the result.

8)70(.875
64
60
56
40
40

Ex. 2. Reduce $\frac{5}{827}$ to a decimal.

827)5000(.00604594
4962
3800
3308
4920
4135
7850
7443
4070
3308

In this example the numerator must be reduced to *thousandths* before the division can take place, consequently the first figure of the quotient will be *thousandths*, and will therefore stand in the third place from the decimal point; two ciphers must therefore be prefixed, in order that the digit may occupy its proper rank. If the ciphers be considered as decimals, the quotient may be pointed by the general rule; there are eight ciphers annexed, that is, there are eight decimals in the dividend, now there are none in the divisor, hence there must be eight in the quotient, that the number in the divisor and quotient together may be equal to the number in the dividend; two ciphers must therefore be prefixed to make up the number. (No. 69.)

The fractional value $\frac{5}{827}$ cannot be expressed exactly by means of decimals; to the extent, however, to which we have carried the approximation, it does not differ from the true value, by so much as the hundred millionth part of a unit, which is more than sufficient for most practical purposes.

If the succeeding part of the decimal be greater than the half of a unit in that place to which the approximation is carried, a more correct value will be obtained by increasing the figure in that place by one. Thus, in the preceding example, were the approximation to be carried only to 5 decimal places, we would increase the figure 4 in the fifth place by 1, and reject all that follow.

72. There are some vulgar fractions which, when reduced to the decimal form, are expressed by a constant recurrence of the same figure or figures. These decimals are, in consequence, called *recurring* or *circulating* decimals, and many tedious and complicated rules have been formed for the management of such fractions. These rules are, however, quite unnecessary, because, when greater exactness is required than can be obtained in the ordinary way, we can reduce the decimal to a vulgar fraction.

Fractions which have any number of *nines* for their denominator, have only, in their decimal form, a repetition of the digit 1.

$$\text{Thus}\quad \frac{1}{9} = 0.1111\text{-----------}$$

$$\frac{1}{99} = 0.010101\text{-------}$$

$$\frac{1}{999} = 0.001001001\text{--}$$

From this periodic decimal fraction we can easily pass to the vulgar fraction from which it is derived. We see, for example, that 0.3333----is equal to 0.1111----multiplied by 3, and as this last is the developement of $\frac{1}{9}$, we conclude that 0.3333----is that of $\frac{1}{9}$ multiplied by 3, or $\frac{3}{9}$ or $\frac{1}{3}$. Again, suppose we had the circulating decimal fraction 0.4545——when there is a constant return of the same two figures 45. This is evidently equal to 0.0101—— multiplied by 45; now, as 0.0101----is the decimal expression of $\frac{1}{99}$ we infer that 0.4545----is that of $\frac{45}{99}$ or of $\frac{15}{33}$ or $\frac{5}{11}$. In the same manner we might show that 0.342342, for example, is equal to $\frac{342}{999}$ or $\frac{38}{111}$, and so on, whatever were the number of circulating figures.

From what has now been stated, we are entitled to infer, that a periodic decimal fraction may be reduced to a vulgar fraction, by taking for the numerator one of these periods, and for the denominator as many nines as the period contains figures. Thus $0.777\text{---} = \frac{7}{9}$; $0.8181\text{---} = \frac{81}{99} = \frac{9}{11}$; $0.135135\text{--} = \frac{135}{999} = \frac{5}{37}$.

73. If the decimal consist of a finite, together with a circulating part, such as $0.42\dot{3}\dot{6}$, where the 36 only repeat, then it may be reduced to a vulgar fraction, by subtracting the finite part from the whole decimal, for the numerator, and by annexing to as many *nines* as there are figures in the circulating part, as many ciphers as there are figures in the finite part, for the denominator, and then reducing the fraction to its lowest terms.

$$\text{Thus}\, 0.42\dot{3}\dot{6} = \frac{4236-42}{9900} = \frac{4194}{9900} = \frac{2097}{4950}.$$

The truth of the rule may be shown as follows.— $0.42\dot{3}\dot{6} = \frac{42.\dot{3}\dot{6}}{100}$ (multiplying and dividing it by 100) =

$\frac{42\frac{56}{99}}{100}$ (No. 72) $=\frac{4200-42+56}{9900}$ (Nos. 26, 46) $=\frac{4200+36-42}{9900}$ $=\frac{4236-42}{9900}$, which brings it to the rule as above stated, and the same may be shown of every fraction of this kind.

74. It may be worth while, before taking leave of this subject, to show the method of reducing quantities of lower denominations to equivalent decimals of a higher denomination, and conversely.

The nature of these processes will be readily understood from examples.

Let it be required to reduce 17s. 6d. to the decimal of a £. This may obviously be done by reducing 17s. 6d. to the fraction of a £, and then that fraction to a decimal. Thus 17s. 6d. $=\frac{7}{8}$ (No. 49.) = 0.875 (No. 71.) Or it may be done as follows: 6d. $=\frac{6}{12}$ of a shilling $=.5$, consequently 17s. 6d. = 17.5, which, when divided by 20 to reduce it to £, will become 0.875.

Ex. 2. Reduce 14s. 9¾d. to the decimal of a £.

The ¾ are first divided by 4 to reduce them to the decimal of a shilling, which is .75; the 9d. is then prefixed, after which the whole is divided by 12, to reduce it to the decimal of a shilling; lastly, the 14 are prefixed, and the whole divided by 20, to reduce it to the decimal of a £.

$$\begin{array}{r|l} 4 & 3.00 \\ \hline 12 & 9.75 \\ \hline 20 & 14.8125 \\ \hline & .740625 \end{array}$$

Conversely, given a decimal to find its value.

Required the value of £ 478125.

$.478125=\frac{478125}{1000000}$, now the numerator will be multiplied by 20, to reduce the pounds to shillings, after which the product must be divided by the denominator, which will be done by pointing off 6 figures, since there are 6 ciphers in the denominator, and so on to the lowest denomination.

$$\begin{array}{r} .478125 \\ 20 \\ \hline 9.562500 \\ 12 \\ \hline 6.750000 \\ 4 \\ \hline 3.000000 \end{array}$$

EXERCISES IN DECIMAL FRACTIONS.

1. Reduce $\frac{15}{16}$ to a decimal fraction. Ans. 0.9375.

2. Reduce to decimals, and then add together the fractions $\frac{4}{5}$, $\frac{7}{8}$, $\frac{3}{40}$, and $\frac{3}{500}$. Ans. 1.756.

3. From $\frac{11}{20}$ take $\frac{21}{40}$ after reducing them to decimals. Ans. 0.025.

4. Multiply $\frac{3}{80}$ by $\frac{9}{400}$ after reducing them to decimals. Ans. 0.00084375.

5. Divide $\frac{3}{4}$ by $7\frac{1}{2}$, after reducing them to decimals.
Ans. 0.1.

6. Reduce 0.625 to a vulgar fraction. Ans. $\frac{5}{8}$.

7. Reduce 0.0015 to a vulgar fraction. Ans. $\frac{3}{2000}$.

8. Reduce 0.666, &c. to a vulgar fraction. Ans. $\frac{2}{3}$.

9. Reduce $0.27\dot{6}\dot{3}$ to a vulgar fraction. Ans. $\frac{76}{275}$.

10. Reduce 15s. $6\frac{3}{4}$d. to the decimal of a pound.
Ans. 0.778125.

11. Reduce $6\frac{3}{4}$d. to the decimal of a shilling. Ans. 0.5625.

12. Required the value of 0.678125 of a pound.
Ans. 13s. $6\frac{3}{4}$d.

PART II.

ALGEBRA.

1. ALGEBRA is that branch of Mathematical Science in which quantity is made the subject of calculation by means of certain general signs and symbols. Algebra differs from Arithmetic chiefly in this; that, in the latter, every figure has a determinate and individual value peculiar to itself; whereas, in Algebra, the characters which are employed to denote quantities are general, or independant of any particular signification, and may, therefore, represent all sorts of numbers or quantities, according to the nature of the question to which they are applied.

2. In questions connected with this science, the object is to discover the value of *unknown* quantities, from having their relation specified to such as are *given* or *known*. And, with a view to a more complete distinction, known or given quantities are represented by the first letters of the alphabet, a, b, c, d, &c.; and those which are unknown by the last, x, y, z, &c.

3. To express the relations which quantities bear to each other in respect of composition, certain signs are employed, which are as follow:—

4. The sign $+$ (*plus*) signifies that the quantity to which it is prefixed is to be added. Thus $a+b$ signifies that the quantity represented by b is to be added to that represented by a. If a represent 5 and b 4, then will $a+b$ represent $5+4$ or 9. If no sign is prefixed to a quantity, the sign $+$ is understood; thus a signifies $+a$.

5. The sign $-$ (*minus*) signifies that the quantity before

which it is placed must be subtracted. Thus $a - b$ signifies that b is to be taken from a; if a be 12, and b, 7, then $a - b$ will express $12 - 7$, or 5.

6. Quantities to which the sign + is prefixed, are called *positive* or *affirmative* quantities; and those having the sign — prefixed, are called *negative* quantities. Instead of positive and negative, the terms *additive* and *subtractive* are sometimes employed. Thus, in the expression $a x - y + b - z$, $a x$ and b are the positive or additive quantities, and $-y - z$ the negative or subtractive.

7. The sign × (*into*) signifies that the quantities between which it stands are to be *multiplied* together. Thus, $a \times b$ signifies that the quantity represented by a is to be multiplied by that represented by b; if a be 8 and b, 7, then $a \times b$ will be 8×7, or 56.

A point is sometimes interposed between the factors instead of the sign ×; thus $a . b$ signifies $a \times b$. Both of these signs, however, are frequently omitted; and then the product is indicated by the mere *juxtaposition* of the quantities. Thus, the product of a by b is expressed by $a b$; of a, b and x by $a b x$, &c.

8. The sign ÷ (*divided by*) placed between two quantities, signifies that the former of those quantities is to be divided by the latter. Thus, $a \div b$ signifies that the quantity represented by a is to be divided by that represented by b; if a be 18, and b, 6, then $a \div b$ will be equal to $18 \div 6$, or 3. This sign is seldom employed. The quotient arising from the division of one quantity by another, is indicated by placing the dividend above the divisor, in the form of a fraction, thus, $\frac{a}{b}$ signifies the quotient of a divided by b.

9. The sign = (*equal to*) signifies that the quantities between which it is placed are equal to one another; thus, $a = b$ signifies that the quantity represented by a is equal to that represented by b.

10. The *coefficient* of a quantity is that which multiplies it; thus, in the expressions $5 x$ and $a y$, 5 and a are respectively the coefficients of x and y. And if a quantity have no coefficient expressed, it is understood to be unity; thus x is the same as $1 x$.

11. When a quantity a is multiplied any number of times by itself, the products are called *powers* of the original quantity a, which, in reference to them, is denominated the *root*. If the quantity be multiplied once by itself, the product is called the *square*, or second power; if twice, the *cube*, or third

power, &c. The number of factors which enter into the power is indicated by a small figure on the right of the quantity. Thus the product of $a \times a$ or $a\,a$ is expressed by a^2.
$a \times a \times a$ or $a\,a\,a$ a^3.
In like manner, of $a \times a \times a \times a$ or $a\,a\,a\,a$ by a^4.
The small figure that indicates the number of factors of which the power is composed, is called the *index* or *exponent*. The student must be careful to distinguish between the terms *coefficient* and *index*, or *exponent*. In the expression $3\,x^5$, for example, 3 is the coefficient, and 5 the index, or exponent. In ax^n, a is the coefficient, and n the index or exponent.

12. The roots of quantities are expressed by the sign $\sqrt{}$, together with the proper index, thus:

$\sqrt[2]{a}$ or $\sqrt{a}$ expresses the *square root* of a, or such a quantity as, when multiplied *once* by itself, will produce a.

$\sqrt[3]{a}$ expresses the *cube root* of a, or such a quantity as, when multiplied *twice* into itself, will produce a.

In the same manner, $\sqrt[4]{a+x}$ expresses the fourth root of $a + x$, &c.

13. A *simple* quantity is that which consists of a single term, as a, $b\,x\,y$, $\frac{c}{d}$, &c. A compound quantity is that which consists of two or more terms; if it consist of two, it is called a *binomial*; if of three, a *trinomial*, &c. Thus, $a+b$ is a binomial; $a+b\,x-c$, a trinomial. That particular species of binomial which expresses the difference between two quantities, is called a residual, as $a - b$.

14. When a line is drawn over any number of algebraic quantities, it indicates that they are to be taken *collectively*, or as one whole quantity. Thus $\overline{a+b} \times \overline{c+d}$, means that the sum of a and b is to be multiplied by the sum of c and d
If the quantities stood thus, $\overline{a+b} \times c + d$, then this signifies that the sum of a and b is to be multiplied by c; and that d is to be added to the product. The line drawn over the quantities is called a *vinculum*.

Instead of the *vinculum*, a *parenthesis* is frequently used. Thus, $(a-b)\ (c-d,)$ means that the difference of a and b is to be multiplied by the difference of c and d; $(a+b) \div (c-d)$ signifies that the sum of a and b is to be divided by the difference of c and d.

15. *Like* quantities are those which consist of the *same* letter, or combination of letters; thus, a, $2\,a$, $7\,a$, $3\,a\,b$, $5\,a\,b$, $a\,b\,x^2$, $3\,a\,b\,x^2$; &c. are called *like* quantities. *Unlike* quan-

tities are such as consist of *different* letters, or *different* combinations of letters; thus, $3\,a$, $5\,b$; x^2y, $x\,y^2$ are *unlike* quantities.

16. In order that the student may acquire accurate conceptions of the characters and signs employed in algebra, it will be of advantage to him to compute the numerical values of the following expressions:—

Let $\left.\begin{array}{l} a=9 \\ b=8 \\ c=6 \\ d=5 \\ e=4 \end{array}\right\}$ Then

1. $a+b-e=9+8-4=13$

2. $a\,b+\frac{a+c}{d}=9\times 8+\frac{9+6}{5}=72+3=75$

3. $(a+d)\,(b-e)=(9+5)\,(8-4)=14\times 4=56$

4. $(d\,e+c)\,a+b=(5\times 4+6)9+8=234+8=242$

5. $\frac{a^2+b^2}{d+e}=\frac{81+64}{5+4}=\frac{145}{9}=16\frac{1}{9}$

6. $\sqrt{bc+1}\,\sqrt{ac+2d}=\sqrt{48+1}\,\sqrt{54+10}=\sqrt{49}\,\sqrt{64}=7\times 8=56$

7. $\frac{(2a^2+\sqrt{bd-e})a-d}{\sqrt{3bc}+\sqrt{ab+d}+e+3}=\frac{(162+6)9-5}{12+9+3}=\frac{1512-5}{24}=\frac{1507}{24}=62\frac{19}{24}.$

Required the values of the following expressions.

8. $a^2+3\,a\,b-2\,c\,d$ Ans. 237.

9. $\frac{(a+b)\,(c+d)}{(a+c)\,(b-d)}$ Ans. $4\frac{7}{45}$.

10. $\frac{5\,a^2\,b+2\,a\,b^2}{\sqrt{b\,d+a}+\sqrt{b\,e+4}}$ Ans. $337\frac{11}{13}$.

11. $\frac{5\sqrt{a\,e}+3\sqrt{5\,c+6}-\frac{1}{2}\sqrt{e}}{\sqrt{2\,c^2+a}-e\sqrt{d^2+e^2+b}+3\,b}$ Ans. $9\frac{2}{5}$.

12. $\frac{(d^2+c\,d)\sqrt{b\,d+a\,e+d}}{\frac{1}{3}(a^2+b^2-c^2)+5\sqrt{a-d}}$ Ans. $15\frac{30}{53}$.

ADDITION.

17 The term *Addition*, as used in Algebra, has a more extended sense than it has in Arithmetic; for here, both Addition and Subtraction, strictly so called, are employed in ding the *sum*.

From the division of algebraic quantities into *positive* and *negative*, *like* and *unlike*, then arise three cases of addition.

CASE I.

To add like quantities with like signs.

18. RULE. *Add together the coefficients of the several quantities, prefix the common sign, and subjoin the common letter or letters.*

The reason of the rule will be readily perceived by attending to the following examples:

Ex. 1. $a+a+a+a+a=a\times 5$ or $5a$. This is obvious, for if a be added five times in *succession*, it is evidently the same thing to add $5a$ at *once*.

Ex. 2. $-b-2b-3b$. The meaning of this expression is, that b is to be subtracted; from the remainder, $2b$ is to be subtracted; and again, from this last remainder, that $3b$ is to be subtracted. Now, the performing of these successive subtractions, will evidently lead to the same result, as the subtraction of $6b$; and in order to indicate that $6b$ is *subtractive*, or is to be subtracted, the sign $-$ is prefixed thus, $-6b$.

$$\begin{array}{rrr}
\text{Ex. 3.} & \text{Ex. 4.} & \text{Ex. 5.} \\
3a+\ 2b & 3x^2-\ 4xy & a^2x^2+2xy^2-\ 4x^2y \\
4a+\ 5b & x^2-\ 3xy & 2a^2x^2+\ xy^2-\ 3x^2y \\
7a+\ b & 5x^2-\ 5xy & 5a^2x^2+3xy^2-\ 5x^2y \\
a+\ 8b & 7x^2-\ 8xy & 11a^2x^2+2xy^2-12x^2y \\
\hline
15a+16b & 16x^2-17xy & 19a^2x^2+8xy^2-24x^2y
\end{array}$$

In these examples, it may be observed, that unity is understood to be coefficient of those quantities, which have none expressed. Thus, in adding up the first column of Ex. 3, we have $1+7+4+3=15$. In the first column of Ex. 5, $11+5+2+1=19$; and so of the rest.

The following are intended as exercises to the student; the answers are not subjoined, but the correctness of the process can be proved as in addition of arithmetic.

$$\begin{array}{rrr}
\text{Ex. 6.} & \text{Ex. 7.} & \text{Ex. 8.} \\
4x^3+\ x^2-\ xy & 3x^2y^3-4a^2x^3+\ x & x^n-7x^2y+4 \\
x^3+2x^2-3xy & x^2y^3-\ a^2x^3+3x & 2x^n-\ x^2y+1 \\
5x^3+4x^2-\ 5xy & 5x^2y^3-5a^2x^3+5x & 3x^n-4x^2y+3 \\
3x^3+3x^2-8xy & 8x^2y^3-7a^2x^3+4x & 5x^n-2x^2y+5 \\
\hline
 & & \\
\hline\hline
\end{array}$$

CASE II.

To add like quantities with unlike signs.

19. **Rule.** *Collect the coefficients of the positive quantities into one sum, and those of the negative into another; subtract the less of these sums from the greater, prefix the sign of the greater, and subjoin the common letter or letters.*

The reason of this rule is very evident. Suppose we have to add the quantities $7a$ and $-3a$ together. The meaning of this expression is, that $7a$ is to be added, and $3a$ to be subtracted; now, to add $7a$, and then to subtract $3a$ is manifestly the same thing as to add $4a$; *that is* the sum of $7a$ and $-3a$ is $+4a$. In the same manner, if we have $-7a$ and $+3a$, the sum will be $-4a$, because to subtract $7a$, and then add $3a$, amounts to the same thing as to subtract $4a$.

Ex. 1.	Ex. 2.	Ex. 3.
$3xy+4x^2-3$	$ab-3ax+2z$	$2axy-8+7xy^2$
$-xy-3x^2+7$	$4ab+ax-5z$	$-axy+7-2xy^2$
$5xy+8x^2-2$	$-2ab-5ax+3z$	$5axy-4+5xy^2$
$7xy-7x^2-9$	$5ab+2ax-7z$	$-3axy-5-10xy^2$
$14xy+2x^2-7$	$8ab-5ax-7z$	$3axy-10$ *

If the sum of the positive terms be equal to that of the negative ones, their difference will of course be 0; and hence the total sum will be 0, as in the third column of Ex. 3.

Ex. 4.	Ex. 5.	Ex. 6.
$5x^3-3xy+4a^2x$	$4a^2y-7x^2+xy$	$2xy^2-3ax+5$
$-4x^3+4xy-2a^2x$	$-2a^2y+8x^2+3xy$	$-xy^2+7ax-2$
$8x^3-7xy+5a^2x$	$5a^2y+x^2-5xy$	$3xy^2-5ax+4$
$2x^3+2xy-6a^2x$	$-3a^2y-2x^2-7xy$	$-4xy^2+2ax-8$

CASE III.

To add unlike quantities.

20. **Rule.** *Collect the quantities into one line, prefixing their proper signs.*

Ex. 1. Required the sum of the quantities a, b, c and $-d$. Their sum $=a+b+c-d$.

Ex. 2. Required the sum of the quantities $2a$, $-3b$, x, $7a$ and $2b$. Their sum $=2a-3b+x+7a+2b=9a-b+x$. In this last example, the quantities $2a$, $7a$, as also $-3b$ and $2b$, can be *condensed* or formed into one, being like.

	Ex. 3.		Ex. 4.
	$4x^2-3y+7$		$3ax-7ay+xy^2$
	$2xy+4y^2-a$		$5a^2x+3x^2y+2a^2$
Sum,	$4x^2-3y+7+2xy+4y^2-a$	Sum,	$3ax-7ay+xy^2+5a^2x+$ $3x^2y+2a^2$

SUBTRACTION.

21. The term *Subtraction* is used in Algebra, in a sense correspondingly enlarged with that of Addition; for here both Addition and Subtraction, properly so called, are employed in finding the *difference*.

Rule. *Change the signs of the quantities to be subtracted, and then proceed as in addition.*

When the sign of the subtrahend is + (*plus*,) the reason of the rule is very evident. Thus, for example, if it were required to subtract $2x$ from $3a$, the difference is properly represented by $3a-2x$, because the sign — (*minus*,) prefixed to the $2x$, shows that it is to be subtracted from $3a$. That the rule leads to the true result, when any of the quantities to be subtracted have the sign — prefixed, may be shown as follows. Let it be required to subtract $2b-3x$ from $5a$. It is obvious that $2b-3x$ is less than $2b$ by $3x$, and, therefore, that if $2b-3x$ were subtracted from $5a$, the remainder would be greater by $3x$, than if $2b$ by itself were subtracted. Now, when $2b$ is subtracted, the remainder is $5a-2b$, and, therefore, when $2b-3x$ is subtracted, the remainder must be $5a-2b+3x$.

The same thing may be shown in the following manner: The difference of the two quantities, $5a$ and $2b-3x$, will not be altered if the same quantity $+3x$ be added to both. Let this be done, then we shall have $5a+3x$ and $2b-3x+3x$, or $5a+3x$ and $2b$, (because $-3x$ and $+3x$ are $=0$;) now, by subtracting $2b$, the difference will be $5a+3x-2b$ or $5a-2b+3x$ as before.

It is hardly necessary to state, that the above is the result obtained by the rule. For, if the signs of the subtrahend be changed, it becomes $-2b+3x$, which, added to the $5a$, gives $5a-2b+3x$.

Ex. 1. From $4ab-3c+4$, take $2x^2-3xy+xy^2$. The subtrahend with its signs changed, is $-2x^2+3xy-xy^2$; and, therefore, the remainder is $4ab-3c+4-2x^2+3xy-xy^2$.

When any of the quantities in the subtrahend are like those in the minuend, it will be more convenient to place the one under the other, *conceiving* the signs of the subtrahend changed, and then proceeding as in addition.

Ex. 2.	Ex. 3.	Ex. 4.
$5x^2y-5ax+2xy^2$	$7a^2+\ 5bc-3z$	$5\sqrt{xy}-4x^n+11$
$3x^2y+4ax-7xy^2$	$2x^2-\ 8bc+5z$	$3\sqrt{xy}-4x^n-\ 3$
$2x^2y-9ax+9xy^2$	$5a^2+13bc-8z$	$2\sqrt{xy}\quad * \quad +14$

EXERCISES.

5. From $8ax-4yz+7x^2$ take $5ax+3yz-8x^2$
6. From $3x^2y^2+4axy-7$ take $x^2y^2-3axy+1$
7. From $a\sqrt{x^2y}-b\sqrt{cd}+xy^2$ take $-3a\sqrt{x^2y}+2b\sqrt{cd}-3xy^2$.
8. From a^n+b^m-c take p^2-q^3+3.

MULTIPLICATION.

22. Multiplication in Algebra, signifies the same thing as it does in Arithmetic. To multiply a by b, for example, signifies that a is to be repeated as often as there are units in b. The product is of course $a\times b$, or simply $a\,b$, (No. 7.)* The rule for the multiplication of algebraic quantities, is commonly divided into three cases, in each of which, it must be observed, that if the signs of the factors be like, *that is*, either both positive +, or both negative —, the sign of the product will be +; but if they be unlike, or the one positive +, and the other negative —, then the sign of the product will be —.

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* When any reference is made without stating to what, Algebra is understood; where Arithmetic is meant, it will be expressed.

The preceding rules for the sign of the product may be explained as follows:

(1*st*), If $+a$ is to be multiplied by $+b$, the meaning is, that $+a$ is to be taken b times, or as often as there are units in b, consequently the product will be $+ab$. (2*d*,) If $-a$ is to be multiplied by $+b$, it means that $-a$ is to be repeated b times, or as often as there are units in b, and, therefore, the product is $-ab$. (3*d*,) If $+a$ is to be multiplied by $-b$, it means that $+a$ is to be subtracted as often as there are units in b, and, consequently, the product is $-ab$. (4*th*,) If $-a$ is to be multiplied by $-b$, it means that $-a$ is to be subtracted as often as there are units in b; and since to subtract a negative quantity $-ab$ is the same as to add a positive one, the product will be $+ab$.

Or these four rules may all be comprehended in *one*, thus: To multiply $a-b$ by $c-d$, is to add $a-b$ to itself as often as there are units in $c-d$; now, this is done by adding it c times, and subtracting in d times.

$$\begin{array}{ll}
\text{But } a-b \textit{ added } c \text{ times} & =ac-bc \\
\text{and } a-b \textit{ subtracted } d \text{ times} & =-ad+bd \\
\hline
\text{therefore, } \overline{a-b}\times\overline{c-d} & =ac-bc-ad+bd
\end{array}$$

$$\begin{aligned}
\text{That is, } +a\times+c&=+ac \\
-b\times+c&=-bc \\
+a\times-d&=-ad \\
-b\times-d&=+bd
\end{aligned}$$

23. Powers of the same quantity are multiplied together, by adding their indices or exponents. Thus, $a^2\times a^3=a^{2+3}=a^5$. For $a^2=aa$ and $a^3=aaa$, (No. 11.) therefore, $a^2\times a^3=aa\times aaa=aaaaa=a^5$. If a is to be multiplied by a^2, the product will be a^3 by the same rule; for $a=a^1$ and $a^1\times a^2=a^{1+2}=a^3$. In the same manner, $a^m\times a^n=a^{m+n}$.

CASE I.

When the Factors are both Simple Quantities.

24. RULE. *Multiply the coefficients of the two terms together; and to the product subjoin all the letters or their powers belonging to each term, in their alphabetical order;*

and the result, with the proper sign prefixed, will be the product required.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$9a$	$+7ab$	$-18ax$	$-5x^2y^3$
$3b$	$-4cd$	$+5ax^2$	$-3xy^2$
$27ab$	$-28abcd$	$-90a^2x^3$	$+15x^3y^5$

The first and second examples are so plain, as to require no explanation. In the third, *a* appears in each factor, which will give aa or a^2 in the product. In the same manner, $x\times x^2=xxx$ or x^3, &c.

Ex. 5.	Ex. 6.	Ex. 7.	Ex. 8.
$5a^2c$	$17a^2bc$	$-14abx^2$	$-9pqr$
$3ac^2$	$-4ab^3c^2$	$-8a^2bx$	$-4p^2q^3r$

CASE II.

When one of the Factors is a Compound Quantity.

25. RULE. *Multiply each term of the compound quantity by the simple factor as in the last case, and the result will be the product required.*

Ex. 1.	Ex. 2.
$3ax-2ay+bx^2$	$4x^3y^2-5xy^2+a^2x^2-4$
$3ab$	$-5xy$
$9a^2bx-6a^2by+3ab^2x^2$	$-20x^4y^3+25x^2y^3-5a^2x^3y+20xy$

Ex. 3.	Ex. 4.
$9x^2-4ax^2+3xy-2$	$12a^2-4+18xy^2-3y$
$5ax$	$-2axy$

CASE III.

When both Factors are Compound Quantities.

26. RULE. *Multiply every term of the multiplicand separately by each term of the multiplier, then the sum of all*

the partial products, will be the total product of the given quantities.

Ex. 1.

$$\begin{array}{lr} & a+b \\ & a+b \\ \hline \text{1st, by } a, & a^2+ab \\ \text{2d, by } b, & ab+b^2 \\ \hline & a^2+2ab+b^2 \end{array}$$

Ex. 2.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab^2-b^2 \\ \hline a^2 \quad * \quad -b^2 \end{array}$$

Ex. 3.

$$\begin{array}{r} a+b \\ c+d \\ \hline ac+bc \qquad \\ ad+bd \\ \hline ac+bc+ad+bd \end{array}$$

Ex. 4.

$$\begin{array}{l} a^2+ab+b^2 \\ \qquad a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \quad * \quad * -b^3 \end{array}$$

Ex. 5.

$$\begin{array}{l} 5x^2+2xy-3 \\ \qquad 2x-4 \\ \hline 10x^3+4x^2y-6x \\ \qquad -20x^2-8xy+12 \\ \hline 10x^3+4x^2y-20x^2-6x-8xy+12 \end{array}$$

27. The product is often merely indicated. Thus, the product of the sum of a and b into c is $(a+b)c$. In many of the operations of algebra it is very convenient to resolve a product into its factors. Thus $a x^2 + b x^2 = (a+b)x^2$; $p x + r x + x = (p+r+1)x$.

Ex. 1. Multiply $a^2 + 2ab + b^2$ by $a+b$
Ans. $a^3 + 3a^2b + 3ab^2 + b^3$.

Ex. 2. Multiply $x^3 - x^2 + x - 5$ by $2x^2 + x + 1$
Ans. $2x^5 - x^4 + 2x^3 - 10x^2 - 4x - 5$.

Ex. 3. Multiply $a^3 + b^3$ by $a^3 - b^3$ Ans. $a^6 - b^6$.

Ex. 4. Multiply $3a^2 - 2ax + 4x^2$ by $5a^2 - 3ax - 2x^2$
Ans. $15a^4 - 19a^3x + 20a^2x^2 - 8ax^3 - 8x^4$.

Ex. 5. Multiply $3x^2 - \frac{2}{3}x + \frac{1}{2}$ by $x - \frac{1}{2}$.
Ans. $3x^3 + \frac{13}{6}x^2 + \frac{5}{6}x - \frac{1}{4}$.

DIVISION.

28. Division in Algebra is performed like that in Arithmetic, and when the division cannot actually be made, it is indicated either by placing the sign $\div$ between the dividend and divisor, or oftener, the latter under the former. Thus, a divided by $b = a \div b$ or $\frac{a}{b}$.

The rule for the sign of the quotient follows readily from that for the product. For, since the product of the quotient and divisor is equal to the dividend, it is manifest that if the signs of the dividend and divisor be *like, that is,* both +, or both —, the sign of the quotient will be +; and if they be unlike, *that is,* the one + and the other —, the sign of the quotient will be —.

The preceding rules may be conveniently exhibited as follows:—

$$\text{If } +a \times +b = +ab, \text{ then } \frac{+ab}{+a} = +b \text{ and } \frac{+ab}{+b} = +a$$

$$+a \times -b = -ab, \text{ then } \frac{-ab}{+a} = -b \text{ and } \frac{-ab}{-b} = +a$$

$$-a \times -b = +ab, \text{ then } \frac{+ab}{-a} = -b \text{ and } \frac{+ab}{-b} = -a$$

Hence we infer, that *like* signs produce + in the quotient, and *unlike* signs —.

29. Powers of the same quantity are divided by subtracting the index of the divisor from that of the dividend. Thus, $\frac{a^5}{a^2} = a^{5-2} = a^3$; in the same manner $\frac{a^m}{a^n} = a^{m-n}$; $\frac{a^m b^n}{a^p b^q} = a^{m-p} b^{n-q}$.

Of Division there are three distinct cases, as in Multiplication.

Case I.

When the dividend and divisor are both simple terms.

30. Rule. *Place the dividend over the divisor, in the manner of a fraction, and then reduce it to its simplest form, by cancelling the letters and figures that are common to both terms.*

EXAMPLES.

$18abc \div 9ac$ or $\frac{18abc}{9ac} = 2b$; $-8a^2x^2y \div 4ax$, or $\frac{-8a^2x^2y}{4ax} = -2axy$; $-6ax^2 \div -8ax$, or $\frac{-6ax^2}{-8ax} = \frac{3}{4}x$; $\frac{a}{a} = 1$; $\frac{-a}{a} = -1$.

Ex. 1. Divide $16 a x^2$ by $8 a x$. Ans. $2 x$.
Ex. 2. Divide $-18 x^3 y^2$ by $9 x y$. Ans. $-2 x^2 y$.
Ex. 3. Divide $-15 x y^3$ by $-5 y$. Ans. $3 x y^2$.

Ex. 4. Divide $10 a^2 x^3 y^2$ by $-12 a x^2 y$. Ans. $-\frac{5}{6} a x y$.

Ex. 5. Divide $8 a^m$ by $10 a^n$. Ans. $\frac{4}{5} a^{m-n}$.

Ex. 6. Divide $\frac{5}{6} a^3 x^2$ by $\frac{2}{3} a x^3$. Ans. $\frac{5 a^2}{4 x}$.

Case II.

31. *When the divisor is a simple quantity, and the dividend a compound one.*

Rule. *Divide each term of the dividend by the divisor, as in the former case, setting down such as will not divide in their simplest form.*

EXAMPLES.

$(a b + b c \div b$ or $\frac{a b + b c}{b} = a + c$; $(6 a x + 8 a y) \div 2 a$ or $\frac{6 a x + 8 a y}{2 a} = 3 x + 4 y$; $(12 a^2 x^2 - 8 a x^3) \div 4 a x$ or $\frac{12 a^2 x^2 - 8 a x^3}{4 a x} = 3 a x - 2 x^2$; $(12 a^3 x^2 y^2 - 10 a x^2 + 8 a^2 x) \div 6 a x^2$ or $\frac{12 a^3 x^2 y^2 - 10 a x^2 + 8 a^2 x}{6 a x^2} = 2 a^2 y^2 - \frac{5}{3} + \frac{4a}{3x}$.

Ex. 1. Divide $6 a b c + 4 a b^2$ by $2 a b$. Ans. $3 c + 2 b$.
Ex. 2. Divide $9 a x^2 + 12 a y^2 - 6 a^2$ by $3a$. Ans. $3x^2 + 4y^2 - 2a$.
Ex. 3. Divide $12 a^2 c x - 10 a c^2 x^2 + 6 a^2$ by $8 a^2$.
Ans. $\frac{3}{2} c x - \frac{5 c^2 x^2}{4 a} + \frac{3}{4}$.

Case III.

32. *When the divisor and dividend are both compound quantities.*

Rule. *Place the divisor and dividend in the same manner as in Arithmetic; arranging the terms of each of them ac-*

cording to the powers of the leading quantity; divide the first term of the dividend by the first term of the divisor, and place the result in the quotient; multiply each term of the divisor by this quantity, placing the product under the like terms of the dividend; subtract it from them, and to the remainder bring down as many terms of the dividend, as will make its number of terms equal to that of the divisor; proceed as before until all the terms of the dividend are brought down as in divison of common Arithmetic.

Ex. 1. Divide $a^2 + 2ab + b^2$ by $a + b$.

$$\begin{array}{rl}
\text{Divisor.} & \text{Dividend.} \qquad \text{Quotient.} \\
a + b) & a^2 + 2ab + b^2 (a + b \\
 & a^2 + ab \\ \hline
 & \qquad ab + b^2 \\
 & \qquad ab + b^2 \\ \hline
 & \qquad * \quad *
\end{array}$$

The first term of the dividend (a^2), divided by the first term of the divisor (a), gives a as a quotient; this multiplied into the divisor $a + b$, gives $a^2 + ab$, which, when subtracted from the corresponding terms ($a^2 + 2ab$,) leaves ab; to this remainder the next term (b^2) of the dividend is brought down; the first term ab of this partial dividend ($ab + b^2$), being divided as before by the first term (a) of the divisor, gives b as a quotient; this, multiplied into the whole divisor ($a+b$), gives $ab + b^2$, which, when subtracted, leaves no remainder; the quotient is therefore $a + b$. In the same way may the process be continued, whatever were the number of terms. If the dividend is not exactly divisible by the divisor, the remainder is put over the divisor in the form of a fraction.

Ex. 2.

$$\begin{array}{rl}
a - b) & a^3 - 3a^2b + 3ab^2 - b^3 (a^2 - 2ab + b^2 \\
 & a^3 - a^2b \\ \hline
 & \quad -2a^2b + 3ab^2 \\
 & \quad -2a^2b + 2ab^2 \\ \hline
 & \qquad\qquad ab^2 - b^3 \\
 & \qquad\qquad ab^2 - b^3 \\ \hline
 & \qquad\qquad * \quad *
\end{array}$$

The quotient will be the same in value, if both divisor and dividend are ranged according to the powers of b.

$$\begin{array}{l}
-b+a\,)-b^3+3ab^2-3a^2b+a^3\,(\,+b^2-2ab+a^2 \\
\qquad\qquad \underline{-b^3+\ ab^2} \\
\qquad\qquad\quad 2ab^2-3a^2b \\
\qquad\qquad\quad \underline{2ab^2-2a^2b} \\
\qquad\qquad\qquad\quad -a^2b+a^3 \\
\qquad\qquad\qquad\quad \underline{-a^2b+a^3} \\
\qquad\qquad\qquad\qquad * \quad *
\end{array}$$

The quotient would even be the same in value, in whatever order the terms were ranged; but in no way could it be so simply expressed as it is in the two preceding examples.

Ex. 3. Divide $a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$ by $a^2+2ax+x^2$.

$$\begin{array}{l}
a^2+2ax+x^2)a^5+5a^4x+10a^3x10^2+a^2x^3+5ax^4+x^5(a^3+3a^2x+3ax^2+x^3. \\
\qquad\qquad\quad \underline{a^5+2a^4x+\quad a^3x^2} \\
\qquad\qquad\qquad\quad 3a^4x+\ 9a^3x^2+10a^2x^3 \\
\qquad\qquad\qquad\quad \underline{3a^4x+\ 6a^3x^2+\ 3a^2x^3} \\
\qquad\qquad\qquad\qquad\quad 3a^3x^2+\ 7a^2x^3+5ax^4 \\
\qquad\qquad\qquad\qquad\quad \underline{3a^3x^2+\ 6a^2x^3+3ax^4} \\
\qquad\qquad\qquad\qquad\qquad\quad a^2x^3+2ax^4+x^5 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{a^2x^3+2ax^4+x^5} \\
\qquad\qquad\qquad\qquad\qquad\qquad * \quad * \quad *
\end{array}$$

Ex. 4. Divide a^5-x^5 by $a-x$.

$$\begin{array}{l}
a-x)a^5-\ x^5(a^4+a^3x+a^2x^2+ax^3+x^4 \\
\qquad\ \ \underline{a^5\ -a^4x} \\
\qquad\qquad a^4x-x^5 \\
\qquad\qquad \underline{a^4x-a^3x^2} \\
\qquad\qquad\quad a^3x^2-x^5 \\
\qquad\qquad\quad \underline{a^3x^2-a^2x^3} \\
\qquad\qquad\qquad a^2x^3-x^5 \\
\qquad\qquad\qquad \underline{a^2x^3-ax^4} \\
\qquad\qquad\qquad\quad ax^4-x^5 \\
\qquad\qquad\qquad\quad \underline{ax^4-x^5} \\
\qquad\qquad\qquad\qquad * \quad *
\end{array}$$

Ex. 5. Divide $8x^3-18x^2+30x-20$ by $4x-3$.

$$4x-3)8x^3-18x^2+30x-20(2x^2-3x+5+\frac{x-5}{4x-3}$$

$$\begin{array}{r}
8x^3-6x^2 \\
\hline
-12x^2+30x \\
-12x^2+9x \\
\hline
21x-20 \\
20x-15 \\
\hline
x-5
\end{array}$$

33. The division of Algebraic quantities, may sometimes be carried on *ad infinitum*, like a recurring decimal. A few terms, however, of the quotient, will be sufficient to indicate the rest; and, therefore, supersede the necessity of actually going through the division.

Ex. 6. Divide a by $a+x$.

$$a+x)a\text{----}(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+, \text{\&c.}$$

$$\begin{array}{l}
a+x \\
\hline
-x \\
-x-\frac{x^2}{a} \\
\hline
\qquad\frac{x^2}{a} \\
\qquad\frac{x^2}{a}+\frac{x^3}{a^2} \\
\hline
\qquad\quad-\frac{x^3}{a^2} \\
\qquad\quad-\frac{x^3}{a^2}-\frac{x^4}{a^3} \\
\hline
\qquad\qquad+\text{\&c.}
\end{array}$$

In the same manner, it may be shown, that $\frac{a}{a-x}=1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+$, &c. where the *law* by which either of these series may be continued, is obvious.

EXAMPLES.

Ex. 1. Divide $a^2-2ab+b^2$ by $a-b$. Ans. $a-b$.

Ex. 2. Divide a^2-b^2 by $a-b$. Ans. $a+b$.

Ex. 3. Divide $9x^3+3x^2-38x+24$ by $3x-4$. Ans. $3x^2+5x-6$.

Ex. 4. Divide $2x^2y^2-x^3y+4xy-6x^4+20x^2-16$ by $xy-2x^2+4$. Ans. $2xy+3x^2-4$.

Ex. 5. Divide a^6-x^6 by $a-x$. Ans. $a^5+a^4x+a^3x^2+a^2x^3+ax^1+x^5$.

Ex. 6. Divide $\frac{1}{3}x^3-\frac{31}{40}x^2+\frac{61}{80}x-\frac{3}{8}$ by $\frac{1}{2}x-\frac{3}{5}$.

Ans. $\frac{2}{3}x^2-\frac{3}{4}x+\frac{5}{8}$.

34. *On the application of the preceding rules to quantities with literal coefficients.*

It has been already stated, (No. 27.) that a compound quantity may be expressed by placing the coefficients of like quantities, one after another, in a parenthesis, and then annexing the common letter by which each of the quantities within the parenthesis is to be multiplied. Thus, $ax^3-bx^3+px^3=(a-b+p)x^3$; $-ax^2-bx^2-cx^2=-(a+b+c)x^2$; $-ax^2+bx^2+cx^2=-(a-b-c)x^2$. The first example is very simple; in the second, the terms are all negative, *that is*, are all to be subtracted; and to subtract each of them in succession, amounts to the same thing as to subtract their sum at once; now their sum is $(a+b+c)x^2$, and to show that it is *subtractive*, the sign *minus* is prefixed, thus, $-(a+b+c)x^2$. In the third example, the whole quantity will either be negative or positive, according as ax^2 is greater or less than the sum of bx^2 and cx^2. In either case, the result is properly indicated by the expression $-(a-b-c)x^2$; for this implies that both b and c are to be subtracted from a, the remainder multiplied by x^2, and then this product subtracted; now, if a be greater than the sum of b and c, $a-b-c$ will be *positive*, and, consequently, also positive when multiplied by x^2; a positive quantity is therefore to be subtracted, (as indicated by the sign $-$ prefixed to the whole quantity,) and therefore it will become negative.

On the contrary, if the sum of b and c be greater than a, then $a-b-c$ will be negative, and, consequently, also negative when multiplied by x^2; a negative quantity is therefore to be subtracted, and hence it will become positive.

Ex. 1.

$$\text{Add}\left\{\begin{array}{r} a x^3 - b x^2 + x \\ - c x^3 + d x^2 - r x \\ p x^3 - a x^2 + q x \\ \hline \end{array}\right.$$

$$(a - c + p)x^3 - (b - d + a)x^2 + (1 - r + q)x$$

Ex. 2.

$$\text{Subtract,}\left\{\begin{array}{l} p x^3 - q x^2 + r x \\ m x^3 + n x^2 - s x \\ \hline \end{array}\right.$$

$$(p - m)x^3 - (q + n)x^2 + (r + s)x$$

Ex. 3.

$$\begin{array}{lrrrr} \text{Multiply} & m x^2 & - n x & + r & \\ \text{by} & & p x & - q & \\ \hline & m p x^3 & - n p x^2 & + p r x & \\ & & - m q x^2 & + n q x & - r q \\ \hline \end{array}$$

$$m p x^3 - (n + m)x^2 + (p r + n q)x - r q$$

Ex. 4.

$$\begin{array}{l} x^2 - a x - b\,)\, a x^3 - (a^2 + b)x^2 + b^2 (a x - b \\ \qquad\qquad\quad a x^3 - a^2 x^2 - a b x \\ \hline \qquad\qquad\qquad - b x^2 + a b x + b^2 \\ \qquad\qquad\qquad - b x^2 + a b x + b^2 \\ \hline \end{array}$$

* * *

35. *An example of the application of the elementary rules in deducing a general theorem.*

Let a and b be any two numbers, of which a is the greater, and b the less; let their sum be represented by s, and their difference by d.

$$\begin{array}{l} \text{Then } a + b = s \\ \text{and } \quad a - b = d \\ \hline \end{array}$$

$$\left.\begin{array}{ll} \text{by Addition,} & 2a = s + d \\ \text{and hence,} & a = \frac{s}{2} + \frac{d}{2} \end{array}\right\}$$

$$\left.\begin{array}{ll} \text{by Subtraction,} & 2b = s - d \\ \text{and hence,} & b = \frac{s}{2} - \frac{d}{2} \end{array}\right\}$$

From which we deduce this general theorem, that "if the *sum* and *difference* of any two numbers be given, the *greater* of them may be found, by *adding* half the given difference to half the given sum; and the *less*, by *subtracting* half the given difference from half the given sum."

Ex. The sum of two numbers is 124, and their difference 18, what are the numbers?

$$\frac{1}{2} \text{ the sum} = \frac{124}{2} = 62$$

$$\frac{1}{2} \text{ the diff.} = \frac{18}{2} = \frac{9}{\quad}$$

by Addition, - 71 = the greater number
by Subtraction, - 53 = --- less ---

FRACTIONS.

The rules for the management of Algebraic Fractions, are the same with those in common arithmetic; and, as these have been already explained, it would be superfluous again to detail the principles on which the various rules are founded. An example or two under each, will be sufficient to illustrate the various processes.

36. *To reduce a fraction to its lowest terms.*

Rule. *Find the greatest common measure of the terms of the fraction, as in Arithmetic,* (No. 44.) *and then divide them both by it.*

Ex. Reduce $\frac{x^3 - b^2x}{x^2 + 2bx + b^2}$ to its lowest terms.

$$\begin{array}{r}
x^2+2bx+b^2 \,)\, x^3 - b^2x \quad (\,x \\
\underline{x^3 + 2bx^2 + b^2x} \\
-2bx^2 - 2b^2x
\end{array}$$

$$\begin{array}{r}
{}^{*}\,x+b \,)\, x^2+2bx+b^2 \,(\, x+b \\
\underline{x^2 + bx \quad\;\;} \\
bx + b^2 \\
\underline{bx + b^2} \\
* \quad *
\end{array}$$

---

* In finding the greatest common measure of two quantities, either of them may be multiplied or divided, by any number or quantity,

Hence the greatest common measure of the two terms, is $x+b$, and when they are both divided by it, we obtain $\frac{x^2-bx}{x+b}$ for the given fraction in its lowest terms. The greatest common measure of the terms of a fraction, may often be found by mere inspection.

$$\text{Thus, } \frac{a^2-b^2}{a^2-2ab+b^2}=\frac{(a+b)(a-b)}{(a-b)(a-b)} \text{ and therefore } = \frac{a+b}{a-b}.$$

$$\frac{cx+x^2}{a^2c+a^2x}=\frac{(c+x)x}{(c+x)a^2} \text{ and hence } = \frac{x}{a^2}.$$

EXAMPLES.

Ex. 1. Reduce $\frac{a^2b^2-b^2x}{a^4-x^2}$ to its lowest terms. Ans. $\frac{b^2}{a^2+x}$.

Ex. 2. Reduce $\frac{x^4-b^4}{x^5-b^2x^3}$ to its lowest terms. Ans. $\frac{x^2+b^2}{x^3}$.

Ex. 3. Reduce $\frac{x^3-a^3}{x^4-a^4}$ to its lowest terms.

Ans. $\frac{x^2+ax+a^2}{x^3+ax^2+a^2x+a^3}$.

87. *To reduce a whole number to an equivalent fraction, having any given denominator*, (No. 45. Arith.)

Ex. Reduce $x+2$ to an equivalent fraction, whose denominator shall be $y+3$.

$(x+2)(y+3)=xy+2y+3x+6$, hence $\frac{xy+2y+3x+6}{y+3}=$ fraction sought.

EXAMPLES.

Ex. 1. Reduce $a+b$ to an equivalent fraction, whose denominator shall be $a-b$. Ans. $\frac{a^2-b^2}{a-b}$

which is not a divisor of the other, or that contains no factor which is common to them both, without in any respect changing the result. In the present example, the remainder $-2bx^2-2b^2x$ is divided by $-2bx$, which gives the quotient $x+b$.

Ex. 2. Reduce x^2+xy+y^2 to an equivalent fraction, whose denominator shall be $x-y$. Ans. $\frac{x^3-y^3}{x-y}$.

38. *To reduce a mixed quantity to an improper fraction, and conversely.* (No. 46. Arith.)

Ex. 1. Reduce $2x+\frac{x-2}{3}$ to an improper fraction.

$2x=\frac{2x\times 3}{3}=\frac{6x}{3}$; to this add the fractional part, and we obtain $\frac{6x}{3}+\frac{x-2}{3}=\frac{6x+x-2}{3}=\frac{7x-2}{3}$.

Ex. 2. Reduce $3x-\frac{2x-5}{7}$ to an improper fraction.

$3x=\frac{3x\times 7}{7}=\frac{21x}{7}$; here it must be remarked, that the sign $-$ prefixed to the fraction $\frac{2x-5}{7}$ indicates that the whole of that fraction is to be subtracted, and, consequently, that the signs of each term of the numerator must be *changed* when it is combined with $\frac{21x}{7}$; hence the improper fraction required is

$$\frac{21x-2x+5}{7}=\frac{19x+5}{7}$$

Ex. 3. Reduce $\frac{12ax^2+9a^2x-2y-3}{3ax}$ to a mixed quantity.

$$\frac{12ax^2+9a^2x-2y-3}{3ax}=4x+3a-\frac{2y+3}{3ax}.$$

EXAMPLES.

Ex. 1. Reduce $2a+\frac{a-x}{3}$ to an improper fraction.

Ans. $\frac{7a-x}{x}$.

Ex. 2. Reduce $3x^2-\frac{2x^2-6}{5}$ to an improper fraction.

Ans. $\frac{13x^2+6}{5}$.

Ex. 3. Reduce $a - x + \frac{a^2 - a x}{x}$ to an improper fraction.

Ans. $\frac{a^2 - x^2}{x}$.

Ex. 4. Reduce $2x + 5 - \frac{3x - 6}{9}$ to an improper fraction.

Ans. $\frac{5x + 17}{3}$.

Ex. 5. Reduce $\frac{21x + 14y - 3z}{7}$ to a mixed quantity.

Ans. $3x + 2y - \frac{3z}{7}$.

Ex. 6. Reduce $\frac{11x^2 + 13x - 8}{3}$ to a mixed quantity.

Ans. $3x^2 + 4x - 2 + \frac{2x^2 + x - 2}{3}$.

39. *To reduce a compound fraction to a simple one.* (No. 47. Arith.)

Ex. 1. Reduce $\frac{3}{4}$ of $\frac{8x^2}{9y}$ to a simple fraction.

$$\frac{3}{4} \text{ of } \frac{8x^2}{9y} = \frac{3 \times 8x^2}{4 \times 9y} = \frac{24x^2}{36y} = \frac{2x}{3y}$$

Ex. 2. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{3x^2 - 6x}{8}$ to a simple fraction.

$$\frac{2}{3} \text{ of } \frac{3}{4} \text{ of } \frac{3x^2 - 6x}{8} = \frac{2 \times 3 \times (3x^2 - 6x)}{3 \times 4 \times 8} = \frac{3x^2 - 6x}{16}.$$

EXAMPLES.

Ex. 1. Reduce $\frac{3}{5}$ of $\frac{5x^2 - 10y}{6}$ to a simple fraction.

Ans. $\frac{x^2 - 2y}{2}$.

Ex. 2. Reduce $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{14x^2 + 7x}{10x + 5y}$ to a simple fraction.

Ans. $\frac{2x^2 + x}{2x + y}$.

40. *To reduce fractions of different denominators to other equivalent fractions having the same denominator.* (No. 51. Arith.)

Ex. 1. Reduce $\frac{2x}{3}$, $\frac{3x}{4}$ and $\frac{5x}{6}$ to equivalent fractions having a common denominator.

$\frac{2x}{3} = \frac{2x \times 4 \times 6}{3 \times 4 \times 6} = \frac{48x}{72}$; $\frac{3x}{4} = \frac{3x \times 3 \times 6}{4 \times 3 \times 6} = \frac{54x}{72}$ and $\frac{5x}{6} = \frac{5x \times 3 \times 4}{6 \times 3 + 4} = \frac{60x}{72}$. Hence $\frac{2x}{3}$, $\frac{3x}{4}$ and $\frac{5x}{6}$ are equal to $\frac{48x}{72}$, $\frac{54x}{72}$ and $\frac{60x}{72}$ respectively.

Or by taking the least common multiple (12) of the given denominators,

We shall have $\frac{2x}{3}$, $\frac{3x}{4}$ and $\frac{5x}{6} = \frac{8x}{12}$, $\frac{9x}{12}$ and $\frac{10x}{12}$ respectively.

Ex. 2. Reduce $\frac{3x}{a+x}$, $\frac{a-x}{2x}$ and $\frac{1}{3x}$ to equivalent fractions having a common denominator.

$$\frac{3x}{a+x} = \frac{3x \times 2x \times 3x}{(a+x) \times 2x \times 3x} = \frac{18x^3}{6ax^2 + 6x^3}.$$

$$\frac{a-x}{2x} = \frac{(a-x)(a+x) \times 3x}{2x \times (a+x) \times 3x} = \frac{3a^2x - 3x^3}{6ax^2 + 6x^3}.$$

$$\frac{1}{3x} = \frac{1 \times 2x \times (a+x)}{3x \times 2x \times (a+x)} = \frac{2ax + 2x^2}{6ax^2 + 6x^3}.$$

Hence $\frac{18x^3}{6ax^2 + 6x^3}$, $\frac{3a^2x - 3x^3}{6ax^2 + 6x^3}$ and $\frac{2ax + 2x^2}{6ax^2 + 6x^3}$ are the fractions required.

EXAMPLES.

Ex. 1. Reduce $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ to equivalent fractions having a common denominator.

Ans. $\frac{adf}{bdf}$, $\frac{bcf}{bdf}$ and $\frac{bce}{bdf}$.

Ex. 2. Reduce $\frac{3x^2-2x}{4}$, $\frac{2x+1}{3}$ and $\frac{3x}{5}$ to equivalent fractions having a common denominator.

Ans. $\frac{45x^2-30x}{60}$, $\frac{40x+20}{60}$ and $\frac{36x}{60}$.

Ex. 3. Reduce $\frac{x+1}{x-1}$, $\frac{x-1}{x+1}$ and $\frac{2x}{x^2+1}$ to equivalent fractions having a common denominator.

Ans. $\frac{x^4+2x^3+2x^2+2x+1}{x^4-1}$, $\frac{x^4-2x^3+2x^2-2x+1}{x^4-1}$ and $\frac{2x^3-2x}{x^4-1}$.

ADDITION OF FRACTIONS.

41. *In order that Algebraic Fractions may be added together, they must first be reduced to a common denominator.* (No 52. Arith.)

Ex. 1. Add together the fractions $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$.

$\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ are respectively equivalent to $\frac{6x}{12}$, $\frac{4x}{12}$ and $\frac{3x}{12}$, and $\frac{6x}{12}+\frac{4x}{12}+\frac{3x}{12}=\frac{6x+4x+3x}{12}=\frac{18x}{12}$ the sum required.

Ex. 2. Required the sum of the fractions $\frac{2x-3}{4}$ and $\frac{3x+2}{5}$.

The fractions $\frac{2x-3}{4}$ and $\frac{3x+2}{5}$ when reduced to equivalent fractions having a common denominator, are $\frac{10x-15}{20}$ and $\frac{12x+8}{20}$, the sum of which is equal to $\frac{10x-15}{20}+\frac{12x+8}{20}$ $=\frac{10x-15+12x+8}{20}=\frac{22x-7}{20}$.

EXAMPLES.

Ex. 1. Add $\frac{x-3}{2}$ $\frac{2x+5}{4}$ and $\frac{3x-2}{8}$ together. Ans. $\frac{11x-4}{8}$

Ex. 2. Add $\frac{1}{a+b}$ and $\frac{1}{a-b}$ together. Ans. $\frac{2a}{a^2-b^2}$.

Ex. 3. Add $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ together. Ans. $\frac{2a^2+2b^2}{a^2-b^2}$

Ex. 4. Add $\frac{2x-3}{4}$, $\frac{x-1}{x+1}$ and $\frac{x+1}{x-1}$ together.

Ans. $\frac{2x^3-5x^2-2x+11}{4x^2-4}$

SUBTRACTION OF FRACTIONS.

42. *Proceed as in Arithmetic, (No. 53.)*

Ex. 1. from $\frac{3x}{4}$ take $\frac{2x}{3}$

$$\frac{3x}{4}-\frac{2x}{3}=\frac{9x}{12}-\frac{8x}{12}=\frac{x}{12}$$

Ex. 2. from $\frac{x+3}{x-3}$ take $\frac{x-3}{x+3}$

$$\frac{x+3}{x-3}-\frac{x-3}{x+3}=\frac{x^2+6x+9}{x^2-9}-\frac{x^2-6x+9}{x^2-9}=$$

$$\frac{x^2+6x+9-x^2+6x-9}{x^2-9}=\frac{12x}{x^2-9}$$

EXAMPLES.

Ex. 1. from $\frac{3x+2}{5}$ take $\frac{2x-3}{5x}$ Ans. $\frac{3x^2+3}{5x}$

Ex. 2. from $\frac{1}{a-b}$ take $\frac{1}{a+b}$ Ans. $\frac{2b}{a^2-b^2}$

Ex. 3. from $\frac{3x-5}{7}$ take $\frac{7}{3x-5}$ Ans. $\frac{9x^2-30x+25}{21x-35}$

Ex. 4. from $\frac{m}{n}$ take $\frac{p}{q}$ Ans. $\frac{mq-np}{nq}$

MULTIPLICATION OF FRACTIONS.

43. *Proceed as in Arithmetic, (No. 54.)*

Ex. 1. Multiply $\frac{3x}{4}$ by $\frac{5x}{6}$.

$$\frac{3x}{4}\times\frac{5x}{6}=\frac{3x\times5x}{4\times6}=\frac{15x^2}{24}=\frac{5x^2}{8}$$

Ex. 2. Multiply $\frac{a^2-b^2}{3a}$ by $\frac{2a^2}{a+b}$

From Ex. 2. Case III, of Multiplication $a^2-b^2=(a+b)\ (a-b)$;

hence $\frac{a^2-b^2}{3\,a}\times\frac{2\,a^2}{a+b}=\frac{(a+b)\,(a-b)\times 2\,a^2}{3\,a\,(a+b)}=\frac{(a-b)\,2\,a^2}{3\,a}$

(by dividing both terms by $a+b$) $=\frac{(a-b)\,2\,a}{3}=\frac{2a^2-2\,a\,b}{3}$

EXAMPLES.

Ex. 1. Multiply $\frac{2\,x}{x-3}$ by $\frac{3\,x}{5}$ Ans. $\frac{6\,x^2}{5\,x-15}$

Ex. 2. Multiply $\frac{3\,x^2-2\,x}{x+2}$ by $\frac{3\,x}{2\,x^2-x}$ Ans. $\frac{9\,x^2-6\,x}{2\,x^2+3\,x-2}$

Ex. 3. Multiply $\frac{x^2-4}{5\,x^2}$ by $\frac{3\,x}{x+2}$ Ans. $\frac{3\,x-6}{5\,x}$

Ex. 4. Multiply $\frac{3\,a}{a-b}$ by $\frac{a^2-b^2}{9}$ Ans. $\frac{a^2+a\,b}{3}$

DIVISION OF FRACTIONS.

44. *Proceed as in Arithmetic, (No. 55.)*

Ex. 1. Divide $\frac{8\,x^2}{3}$ by $\frac{4\,x}{5}$

$$\frac{8\,x^2}{3}\div\frac{4\,x}{5}=\frac{8\,x^2}{3}\times\frac{5}{4\,x}=\frac{8\,x^2\times 5}{3\times 4\,x}=\frac{40\,x^2}{12\,x}=\frac{10\,x}{3}$$

Ex. 2. Divide $\frac{4\,x-8}{5}$ by $\frac{2\,x-4}{15}$

$$\frac{4\,x-8}{5}\div\frac{2\,x-4}{15}=\frac{4\,x-8}{5}\times\frac{15}{2\,x-4}=6$$

EXAMPLES.

Ex. 1. Divide $\frac{x^2+5\,x}{x^2-9}$ by $\frac{x^2}{x+3}$ Ans. $\frac{x+5}{x^2-3\,x}$

Ex. 2. Divide $\frac{3\,x+2}{x+5}$ by $\frac{x-5}{3\,x-2}$ Ans. $\frac{6\,x^2-4}{x^2-25}$

Ex. 3. Divide $\frac{2\,a^2-2\,b^2}{3\,a}$ by $\frac{3\,a+3\,b}{4\,b}$ Ans. $\frac{8\,a\,b-8\,b^2}{9\,a}$

Ex. 4. Divide $\frac{a^3-b^3}{x^2-y^2}$ by $\frac{a-b}{x+y}$ Ans. $\frac{a^2+a\,b+b^2}{x-y}$

INVOLUTION AND EVOLUTION.

45. It has been already stated, (No. 11.) that when a quantity is multiplied any number of times by itself, the successive products are called *powers*, of the original quantity, while this quantity itself, in reference to these powers, is denominated the *root*. Thus, the successive powers of a are a, aa, aaa, $aaaa$, &c. or according to the mode of notation explained (No. 11.) a^1, a^2, a^3, a^4, &c., where a^2 and a^3, from geometrical considerations, are respectively called the *square* and *cube* of the given quantity a. The fourth power of a or a^4, is also sometimes called the *biquadrate*; but the higher powers seem to have no other than that of numeral distinction. This process of continued multiplication of a quantity into itself, by which its successive powers are formed, is called *Involution*.

In the powers of Algebraic quantities, the multiplication is merely indicated, but in those of numbers, it is actually performed. Thus, the fifth power of x is expressed by $xxxxx$ or more simply by x^5, when there is a mere indication of the multiplication. The fifth power of 2 is $= 2^5$, by indication, or $2 \times 2 \times 2 \times 2 \times 2 = 32$ by actual multiplication. In the same manner, the fifth power of 3 is $= 3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$, &c. when it must be observed, that the number of factors in the power, is that expressed by the units in its index. In this way the following table, which contains the first six powers of each of the nine digits is formed.

1st	1	2	3	4	5	6	7	8	9
2d	1	4	9	16	25	36	49	64	81
3d	1	8	27	64	125	216	343	512	729
4th	1	16	81	256	625	1296	2401	4096	6561
5th	1	32	243	1024	3125	7776	16807	32768	59049
6th	1	64	729	4096	15625	46656	117649	262144	531441

46. In raising a simple algebraic quantity a to any power, all that is necessary, is to multiply its index by the index of

the power to which it is to be raised. Thus the 5th power of a is $= a^{1\times 5} = a^5$; the 4th power of b^2 is $= b^{2\times 4} = b^8$. This is manifest, for b^2 raised to the fourth power is $= b^2\times b^2\times b^2\times b^2 = b^{2+2+2+2} = b^{2\times 4} = b^8$. In the same manner, a^n raised to the m^{th} power is $a^{n\times m} = a^{mn}$.

The *odd* powers of negative quantities, will have the sign *minus*, and the *even* powers the sign *plus*. Thus, the square of $-a = -a \times -a = +a^2$; the cube of $-a = -a \times -a \times -a = +a^2 \times -a = -a^3$. In the same manner, the fourth power of $-a = -a \times -a \times -a \times -a = -a^3 \times -a = +a^4$. Or in general, $-a^m$ raised to the n^{th} power is $\pm a^{mn}$ according as n is an even or an odd number.

Any power of the product of two or more quantities, is equal to the same power of each of the factors multiplied together: thus, $(a\,b)^2 = a\,b \times a\,b = a^2\,b^2$ or in general $(abc)^m = a^m \times b^m \times c^m$. If the quantity have a numeral coefficient, this coefficient must likewise be raised to the given power, thus $(3\,a^2\,b^3)^4 = 3^4 \times a^{2\times 4} \times b^{3\times 4} = 81a^8b^{12}$.

47. Fractions are raised to any given power, by raising both numerator and denominator to that given power. Thus, the third power of $\frac{3}{4}$ or $\left(\frac{3}{4}\right)^3 = \frac{3^3}{4^3} = \frac{27}{64}$. The fifth power of $\frac{a}{x} = \frac{a^5}{x^5}$. A mixed number or quantity is raised to any power by first reducing it to an improper fraction, and then raising that fraction to the given power. Thus, the third power of $5\frac{3}{4}$ or of $\frac{23}{4} = \frac{23^3}{4^3} = \frac{12167}{64} = 190\frac{7}{64}$; the fourth power of $2\frac{1}{2}$ or of $\frac{5}{2} = \frac{5^4}{2^4} = \frac{625}{16} = 39\frac{1}{16}$.

On the Involution of Compound Algebraic Quantities.

48. The powers of compound algebraic quantities are found by the mere application of the rule for the multiplication of compound quantities, (No. 26.)

Ex. 1. Required the third power of $2a+3b$.

$$\begin{array}{rl}
 & 2a+3b \\
 & 2a+3b \\ \hline
 & 4a^2+6ab \\
 & \qquad\;\; 6ab+9b^2 \\ \hline
\text{Square} = & 4a^2+12ab+9b^2 \\
 & \qquad\qquad\;\; 2a+3b \\ \hline
 & 8a^3+24a^2b+18ab^2 \\
 & \qquad\;\; 12a^2b+36ab^2+27b^3 \\ \hline
\text{Cube} = & 8a^3+36a^2b+44ab^2+27b^3.
\end{array}$$

When the index of the power to which the quantity is to be raised, is a large number, this process of continued multiplication would evidently become extremely tedious and inconvenient. We shall therefore explain the method of finding the powers of compound algebraic quantities by Sir ISAAC NEWTON'S ***Binomial Theorem.*** This theorem admits of a general demonstration, but at present it may be sufficient to deduce it from an induction of a few particular facts.

In the involution of a binomial quantity of the form $a+b$, the powers of each of the letters, together with their coefficients, are found to observe certain laws, which will be readily understood by an examination of the following table :—

TABLE of the Powers of a+b.

Powers.	Mode of expressing them.	Powers Expanded.
1st Power.	$a+b$	$a + b$
2d Power, or Square.	$(a+b)^2$	$a^2 + 2ab + b^2$
3d Power, or Cube.	$(a+b)^3$	$a^3 + 3a^2b + 3ab^2 + b^3$
4th Power.	$(a+b)^4$	$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
5th Power.	$(a+b)^5$	$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
6th Power.	$(a+b)^6$	$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

The successive powers of $a - b$, are precisely the same as those of $a+b$, except that the signs of the terms will be + and — alternately.

49. In reviewing the preceding table, where several of the powers of $a+b$ are expanded, it may be observed, 1*st*, That in any one of those powers, the first term a of the

binomial appears in every term except the last, and the last term b of the binomial in every term except the first. 2*d*, That the index of a, in the first term of the power, is the same with the index of the power to which the binomial is to be raised, and that the indices of the other powers of a, in the successive terms, continually diminish by unity to the last term but one, where it is unity itself. 3*d*, That the index of b, in the second term of the power, is unity, and that the indices of b, in the successive terms, continually increase by unity to the last term, where it is the same with the index of the power to which the binomial is to be raised.

From this statement, it will not be difficult, leaving the coefficients out of the question, to express the literal parts of any power of $a+b$; suppose, for example, the 7th power.

The powers of a and b separately, will be as follows:

$$a^7,\ a^6,\ a^5,\ a^4,\ a^3,\ a^2,\ a$$
$$b,\ b^2,\ b^3,\ b^4,\ b^5,\ b^6,\ b^7$$

Or, combining these into the form of multiplication,

$$a^7,\ a^6b,\ a^5b^2,\ a^4b^3,\ a^3b^4,\ a^2b^5,\ ab^6,\ b^7$$

Next, as to the coefficients of these terms, 1*st*, It may be observed, on examining the table, that the first term which has a coefficient, is the *second* term of the power; and that this coefficient is the same as the index of the first term, or of the power to which the binomial is to be raised. Thus, in the square it is 2; in the cube 3; and in the fourth power 4, &c. 2*d*, That if the coefficient of a in any term be multiplied by its index in that term, and the product divided by the number of terms to that place, reckoning from the left, the quotient will give the coefficient of the next term.

Thus, in the 6th power, $\frac{\text{Coeff. of } a \text{ in the 2d term} \times \text{its index}}{\text{Number of terms to that place}} = \frac{6\times 5}{2} = \frac{30}{2} = 15 =$ coefficient of the third term. In the same manner, $\frac{\text{Coeff. of } a \text{ in the 3d term} \times \text{its index}}{\text{Number of terms to that place}} = \frac{15\times 4}{3} = \frac{60}{3} = 20 =$ coefficient of the fourth term, and so on for the other coefficients. By applying this same rule throughout, the coefficient of the last term will be unity, which is not expressed.

Hence we are furnished with general rules for raising a binomial to any power required, without the laborious process of actual multiplication. Let it be required to find the 7th power of $a+b$. The terms without the coefficients, as already found, are

$$a^7 \quad a^6b \quad a^5b^2 \quad a^4b^3 \quad a^3b^4 \quad a^2b^5 \quad ab^6 \quad b^7$$

The coeff. of the second term is 7; of the third $= \frac{7\times6}{2} = \frac{42}{2} = 21$; of the fourth $= \frac{21\times5}{3} = \frac{105}{3} = 35$; of the fifth $= \frac{35\times4}{4} = 35$; of the sixth $= \frac{35\times3}{5} = \frac{105}{5} = 21$; of the seventh $= \frac{21\times2}{6} = \frac{42}{6} = 7$.

Hence $(a+b)^7 = a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$. In the same manner, it will be found, that $(a-b)^7 = a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$. $(a+b)^{10} = a^{10}+10a^9b+45a^8b^2+120a^7b^3+210a^6b^4+252a^5b^5+210a^4b^6+120a^3b^7+45a^2b^8+10ab^9+b^{10}$.

It may here be remarked, that when the index of the power is an odd number, the number of terms (being always greater by one than the units in its index) will be an even number and conversely. In the former case, the *two* coefficients in the middle will be the same, and those coefficients which are equally distant from the middle ones, will be equal. In the latter case, when the index of the power is an odd number, the greatest coefficient will be in the middle, and will have equal coefficients at equal distances from it. By attending to this law of the coefficients, they need only be computed as far as the middle term or terms, and then the same written in an inverted order.

Thus, the coefficients of $(a+b)^7$ are 7, 21, 35, 35, 21, 7.
$(a+b)^{10}$ are 10, 45, 120, 210, 252, 210, 120, 45, 10.

50. By attending to the rules which have been stated, the binomial theorem may be presented in a more general form. Suppose it were required to raise the binomial

$(a+b)$ to the power whose index is n. Proceeding with n as we have done with the indices in the preceding examples, it appears, that

The first term will be a^n.

The second - - $na^{n-1}b$.

The third - - $\frac{n(n-1)}{2}a^{n-2}b^2$.

The fourth - - $\frac{n(n-1)(n-2)}{2\times 3}a^{n-3}b^3$.

The fifth - - $\frac{n(n-1)(n-2)(n-3)}{2\times 3\times 4}a^{n-4}b^4$.

- - - - - - - - - - - - - - -

The last - - b^n.

$$\text{Or, } (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\times 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2\times 3\times 4}a^{n-4}b^4 + \text{\&c.} \quad - - - - - - - - - - \quad + b^n.$$

The use of this general form of the theorem, will appear in its application to an example or two.

Ex. 1. Required the 4th power of $3x^2+2y$.

In this instance, $n=4$; $a=3x^2$, and $b=2y$, and by substituting these in the general formula, we have,

The 1st term, a^n - - - - - - - - is $(3x^2)^4$ - $=81x^8$.

2d, $na^{n-1}b$ - - - - - - is $4\times(3x^2)^3\times 2y$ - - $=216x^6y$.

3d, $\frac{n\times(n-1)}{2}a^{n-2}b^2$ - is $\frac{4\times 3}{2}\times(3x^2)^2\times(2y)^2=216x^4y^2$.

4th, $\frac{n\times(n-1)(n-2)}{2\times 3}a^{n-3}b^3$ is $\frac{4\times 3\times 2}{2\times 3}\times 3x^2\times(2y)^3=96x^2y^3$.

5th, b^n - - - - - - - - - - - - is $(2y)^4$ - $=16y^4$.

Hence $(3x^2+2y)^4=81x^8+216x^6y+216x^4y^2+96x^2y^3+16y^4$.

Ex. 2. Required the 7th power of $2x-7$.

Hence, $n=7$; $a=2x$, and $b=-7$; and by the formula, $(2x-7)^7=(2x)^7+7\times(2x)^6\times -7+21\times(2x)^5\times -7^2+35\times(2x)^4\times -7^3+35\times(2x)^3\times -7^4+21\times(2x)^2\times -7^5+7\times(2x)\times -7^6-7^7=128x^7-3136x^6+32928x^5-192080x^4+672280x^3-1411788x^2+1647086x-823543$.

51. By means of the binomial theorem, a *trinomial*, or even a *quadrinomial* quantity, may be raised to any power. Let it be required, for example, to find the fourth power of $a+b+c$. By considering $a+b$ as one quantity, we shall have, $(a+b+c)^4 = \overline{(a+b)+c}|^4 = (a+b)^4 + 4(a+b)^3c + 6(a+b)^2c^2 + 4(a+b)c^3+c^4$, as before. Now, expanding these terms by the theorem, we shall have $\overline{a+b+c}|^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4+4a^3c+12a^2bc+12ab^2c+4b^3c+6a^2c^2+12abc^2+6b^2c^2+4ac^3+4bc^3+c^4$. In the same manner, $(a-b+c-d)^3 = \overline{(a-b)+(c-d)}|^3 = (a-b)^3+3(a-b)^2(c-d)+3(a-b)(c-d)^2+(c-d)^3 = a^3-3a^2b+3ab^2-b^3+3a^2c-3a^2d-6abc+6abd+3b^2c-3b^2d+3ac^2-3bc^2-6acd+6bcd+3ad^2-3bd^2+c^3-3c^2d+3cd^2-d^3$.

EXAMPLES.

Ex. 1. Required the third power of $(a-x)$.
Ex. 2. Required the fourth power of $(3y-5)$.
Ex. 3. Required the square of $(a-b+c)$.
Ex. 4. Required the square of $(2x-y+z-3)$.

On the Evolution of Algebraic Quantities.

52. Evolution is the reverse of involution, and therefore explains that process by which, when the *power* is given, the *root* may be found.

It has been already observed, that, when a quantity is multiplied *once* by itself, the product is called the *square* of the given quantity; we may now observe farther, that the quantity itself, in reference to this square, is denominated the *square root*. In the same manner, if a quantity be multiplied *twice* by itself, the product is called the *cube* of the given quantity, which, in reference to this cube, is denominated the *cube root*. The term *biquadrate root*, is sometimes, but seldom used. Higher roots than the third or cube, are generally designated by the corresponding ordinal number, as the *fourth*, *fifth*, &c. *root*.

53. From this, it appears, that to find the square root of a quantity, is nothing else, than to find such a quantity as, when multiplied *once* by itself, will produce the given quantity. Thus, $64 = 8 \times 8$, therefore, the square root of 64 or $\sqrt{64} = 8$; $a^2 = a \times a$, and, therefore, a is the square root of a^2 or $\sqrt{a^2} = a$, where the root may be either positive or negative; because $+ \times +$ gives $+$, and $- \times -$ gives also $+$.

In the same manner, the finding of the cube root consists in finding such a quantity as when multiplied *twice* by itself will produce the given quantity. Thus, $27 = 3 \times 3 \times 3$, therefore, the cube root of 27 or $\sqrt[3]{27} = 3$; $a^3 = a \times a \times a$, and, therefore, $\sqrt[3]{a^3} = a$.

54. The whole statements now made, regarding the extraction of roots, may be presented in a more general and condensed form, by saying, that the finding of *any* root of a quantity, consists in finding that factor, which, when multiplied into itself, once less than the units in the index of the root, will produce the given quantity. Thus, the fifth root of 32 or $\sqrt[5]{32} = 2$, since $32 = 2 \times 2 \times 2 \times 2 \times 2$; the *m*th root of a^m or $\sqrt[m]{a^m} = a$. The cube root of $-x^3$ or $\sqrt[3]{-x^3} = -x$, because $-x^3 = -x \times -x \times -x$. The root of a fraction is found by extracting both the root of its numerator and denominator; thus, $\sqrt{\frac{16}{25}} = \frac{4}{5}$; $\sqrt[3]{\frac{8x^3}{27y^3}} = \frac{2x}{3y}$. The root of a mixed number or quantity, will be found, by first reducing it to an improper fraction, and then extracting; thus, $\sqrt{7\frac{1}{9}} = \sqrt{\frac{64}{9}} = \frac{8}{3} = 2\frac{2}{3}$.

55. If the quantity under the radical sign does not admit of having the required root extracted, it is called a *surd*; thus, $\sqrt{5}$, $\sqrt[3]{9}$, $\sqrt[3]{a^2}$, are surd quantities. The application of the fundamental rules to the management of surds, will be shown afterwards.

56. Any even root of a negative quantity is unassignable. Thus, $\sqrt{-a^2}$, $\sqrt[4]{-a^4}$, &c. cannot be determined, because all the even powers of $+a$ or $-a$ are positive. Quantities, therefore, of the form $\sqrt{-4}$, $\sqrt[4]{-a^4}$, &c. have no real root, and are, therefore, called impossible.

57. A general rule for the extraction of the square root of a compound quantity may be readily obtained, by observing how the terms of the root are to be derived from those of the power. The square of $a+b$, according to the

table (p. 82.) is $a^2+2ab+b^2$, which consists of the squares of the two terms a^2 and b^2, together with twice their product $2ab$. If, therefore, when the terms are ranged according to the powers of a, we extract the square root of the first term (a^2), we will obtain the first term of the root (a); and if the next term ($2ab$) be divided by the double of this part of the root now found, that is, by $2a$, this division will give b the second term of the root. Hence the process will stand as follows:

$$\begin{array}{rl}
 & a^2+2ab+b^2(a+b \\
 & a^2 \\ \hline
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\ \hline
 & * \quad *
\end{array}$$

The square root of the first term a^2, is a, which, being placed in the root, and its square (a^2) subtracted from the corresponding term of the power (a^2), leaves no remainder. The next two terms $2ab$ and b^2 are brought down, the first of which, $2ab$, being divided by $2a$, gives b the second term of the root; and as $2ab+b^2=(2a+b)b$, if to $2a$ the term b be added, and this sum be multiplied by b, the product is $2ab+b^2$, which, when subtracted, leaves no remainder.

The square of $a+b+c$, is (No. 51.) $=a^2+2ab+b^2+2ac+2bc+c^2$, where the root may be derived by a mere continuation of the process, explained in the preceding example.

$$\begin{array}{rl}
 & a^2+2ab+b^2+2ac+2bc+c^2(a+b+c \\
 & a^2 \\ \hline
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\ \hline
2a+2b+c) & 2ac+2bc+c^2 \\
 & 2ac+2bc+c^2 \\ \hline
\end{array}$$

Thus, the two first terms $a+b$ of the root being found as before, the remaining three terms $2ac+2bc+c^2$ of the power are brought down. The part of the root $a+b$, already found, is doubled, which gives $2a+2b$; the first term of the remainder $2ac$ being divided by $2a$ gives c, which, being added to both the divisor and root, and the whole of the former multiplied by it, gives $2ac+2bc+c^2$, and this being subtracted leaves no remainder.

In this manner, the following examples are solved:

Ex. 1. Required the square root of $x^4-\frac{6}{5}x^2+\frac{9}{25}$.

$$\begin{array}{rl}
 & x^4-\frac{6}{5}x^2+\frac{9}{25}\left(x^2-\frac{3}{5}\right. \\
 & x^4 \\ \hline
2x^2-\frac{3}{5}\Big) & -\frac{6}{5}x^2+\frac{9}{25} \\
 & -\frac{6}{5}x^2+\frac{9}{25} \\ \hline
 & \quad \ast \qquad \ast
\end{array}$$

Ex. 2. Required the square root of $4x^2-4xy+y^2+16x-8y+16$.

$$\begin{array}{rl}
 & 4x^2-4xy+y^2+16x-8y+16(2x-y+4 \\
 & 4x^2 \\ \hline
4x-y) & -4xy+y^2 \\
 & -4xy+y^2 \\ \hline
4x-2y+4) & +16x-8y+16 \\
 & +16x-8y+16 \\ \hline
 & \quad \ast \qquad \ast \qquad \ast
\end{array}$$

58. By the same process, we are enabled to discover the square roots of numbers. Let 345 be selected as a root; $345=300+40+5$, now the square of this may be found as that of $a+b+c$, and will be $90000+24000+4600+400+25$, let the square root of this be extracted.

$$\begin{array}{rl}
 & 90000+24000+4600+400+25(300+40+5 \\
 & 90000 \\ \hline
600+40) & 24000+4600+400 \\
 & 24000+1600 \\ \hline
600+80+5) & 3000+400+25 \\
 & 3000+400+25 \\ \hline
 & \quad \ast \qquad \ast \qquad \ast
\end{array}$$

Now, it is manifest that the result will not be altered by collecting those numbers into one sum, which stand in the same line, and leaving out the ciphers which merely serve to indicate the rank of the digits; let this be done, and the above process will assume the following condensed form, from which the following rule is evolved:

$$\begin{array}{r}
11\dot{9}0\dot{2}5(345 \\
9 \quad\quad\;\; \\
\hline
64)290 \quad\; \\
256 \quad\; \\
\hline
685)3425 \\
3425 \\
\hline
\cdots\cdot
\end{array}$$

Divide the number whose root is to be extracted, into periods of two figures each, beginning from the unit's place. Find the greatest number, whose square is contained in the first period, and this will be the first figure of the root. Subtract the square of this figure from the first period, and to the remainder annex the next. Divide this number (omitting the last figure) by twice that part of the root already found, annex the result to the root and also to the divisor; then subtract the product of the divisor thus augmented, by that figure of the root last obtained; and to the remainder annex the next period, with which proceed as before; and so on till the given number be exhausted.

The reason of dividing the number, whose root is to be extracted, into periods of two figures each, is because the square of a number can never have more than twice as many places as its root, and at the least but one less; consequently there will always be just as many figures in the root as there are pairs of figures in the given number, the left hand part consisting in some cases of a half period or one figure.

Ex. 2. Required the square root of 7645225.

$$\begin{array}{r}
7\dot{6}4\dot{5}2\dot{2}5(2765 \\
4 \quad\quad\quad\quad\;\; \\
\hline
47)364 \quad\quad\; \\
329 \quad\quad\; \\
\hline
546)3552 \quad \\
3276 \quad \\
\hline
5525)27625 \\
27625 \\
\hline
\end{array}$$

In extracting the square root of a number, where there is a decimal, the pointing off into periods must proceed both ways from the decimal point; and if, after all the figures of the given number are brought down, there should be a remainder, the process of extraction may be carried on to any degree of accuracy, by continually annexing two ciphers to the remainder.

Ex. 3. Extract the square root of 94876.7542.

$$\begin{array}{rl}
 & 9\dot{4}8\dot{7}6.5\dot{7}8\dot{3}(308.0204+\&c. \\
 & 9 \\
608) & 4876 \\
 & 4864 \\
61602) & 125783 \\
 & 123204 \\
6160404) & 25790000 \\
 & 24641616 \\
 & \&c.\ \&c.
\end{array}$$

59. The rule for the extraction of the cube root of a compound quantity, may be obtained in a similar manner, by observing how the terms of the root are to be derived from those of the power. The cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, which is composed of the cubes of the two terms, together with three times the square of the first into the second, and three times the square of the second into the first. If, therefore, the terms be arranged according to the powers of a, the cube root a of the first term a^3 will be the first term of the root. If the cube of a be subtracted from the given power, there will remain $3a^2b+3ab^2+b^3$; the first term of which, divided by three times the square of a, the part of the root already found, will give b the second term of the root. Now, if to three times the square of the first term, three times the product of the two terms, together with the square of the last term be added, the divisor will be complete. This divisor being multiplied by the last term, and the product subtracted will leave no remainder in this particular example. If there were a remainder, then the other terms of the power would be annexed to it, and the process would proceed as before, until all the terms were exhausted.

Ex. 1. Required the cube root of $a^3+3a^2b+3ab^2+b^3$.

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3(a+b \\
 & a^3 \\ \cline{2-2}
3a^2+3ab+b^2) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\ \cline{2-2}
 & \quad * \quad * \quad *
\end{array}$$

Ex. 2. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{rl}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1(x^2-2x+1 \\
 & x^6 \\ \cline{2-2}
3x^4-6x^3+4x^2) & -6x^5+15x^4-20x^3 \\
 & -6x^5+12x^4-\ 8x^3 \\ \cline{2-2}
3x^4-12x^3+12x^2+3x^2-6x+1 & \big) 3x^4-12x^3+15x^2-6x+1 \\
\text{Or, } 3x^4-12x^3+15x^2-6x+1 & \big) 3x^4-12x^3+15x^2-6x+1 \\ \cline{2-2}
 & \qquad\qquad *
\end{array}$$

60. By a similar process, we are enabled to extract the cube root of any number.

Ex. 1. Let it be required to find the cube root of 16974593.

$$\begin{array}{lr|l}
 & & 16974593(200+50+7 \\
 & & 8000000 \\ \cline{3-3}
3a^2=3\times200^2 & =120000 & 8974593 \\
3ab=3\times200\times50= & 30000 & \\
b^2=50^2 & =\quad 2500 & \\ \hline
3a^2+3ab+b^2 & =152500 & 7625000 \\ \cline{3-3}
3a^2=3\times250^2 & =187500 & 1349593 \\
3ab=3\times250\times7 & =\quad 5250 & \\
b^2=7^2 & =\qquad 49 & \\ \hline
3a^2+3ab+b^2 & =192799 & 1349593 \\ \cline{3-3}
\end{array}$$

By omitting the superfluous ciphers, making a small change upon the factors of the two first of those quantities which form the divisors, and bringing down three figures at a time, the above process will then be changed into the following more convenient form:

$$\begin{array}{r|l}
 & 16\dot{9}74\dot{5}93(257 \\
 & 8 \\
\hline
300\times2^2=1200 & 8974 \\
30\times2\times5=300 & \\
5^2=25 & \\
\hline
1525 & 7625 \\
\hline
300\times25^2=187500 & 1349593 \\
30\times25\times7=5250 & \\
7^2=49 & \\
\hline
192799 & 1349593 \\
\hline
\end{array}$$

As every example is susceptible of a like condensation, the following general rule may be given. Divide the given number into periods of three figures each, beginning from the place of units. Find the cube root (2) of the greatest cube number (8) contained in the first period (16), which will be the first figure of the root. Subtract the cube of this figure from the first period, and to the remainder (8) annex the next, (974). Divide this number by 300 times the square of that figure (2) of the root already found, and the quotient * (5) will be the next figure of the root. To 300 times the square of the first figure (2), add 30 times the product of the *two* figures (2 × 5) of the root now found, together with the square of the latter (5); multiply this sum (1525) by the last figure (5) of the root, and from the dividend (8974) subtract the product (7625). To the remainder (1349) annex the next period (593), and proceed as before.

There are still some considerations, by which the labour prescribed in the preceding rule, may be abridged. Thus, to multiply by 300, is the same as to multiply by 3, and then annex two ciphers; and, by 30, to multiply by 3, and annex one cipher.

Again, referring to the example at the beginning of the present number, it is manifest that the *a* employed in finding the second divisor, is equal to the *a*+*b* employed in finding the first; the $3a^2$, therefore, of the second divisor, will be equal to $3(a+b)^2$ of the first $=3a^2+6ab+3b^2$, which is greater than the first complete divisor $3a^2+3ab+b^2$ by $3ab+2b^2$; if, therefore, $3ab+2b^2$ be added to the first divisor, the sum will be the first part of the succeeding one. This, then, in regard to the numbers, will be found by adding the three last lines together, observing to reckon the middle one twice. Thus, $152500+2\times2500+30000=187500$, the first part of the next divisor; and, in the same way, may the first part of every succeeding divisor be found.

---

* It must here be remarked, that *this* quotient is in general much less than the true quotient, because of the additions which the divisor is afterwards to receive. Thus, 1200 is contained 7 times in 8974, but if 7 were taken in place of 5, it would be found that the resulting quantity, when multiplied by 7, would exceed 8974.

The reason of dividing the given number into periods, of three figures each, is because the cube of a number can never have more than three times as many places as its root, and at the least but two less; consequently, there will always be just as many places in the root, as there are periods of three figures each in the given number, the left hand period consisting in some instances of one or two figures.

By attending to the preceding remarks, the process of the extraction of the cube root of any number may be exhibited in the following condensed form; where, for the sake of conciseness, the sign + is used instead of the word annex.

Ex. 2. Required the cube root of 51097851963712.

		51097851963712(37108
		27
$3^2\times3+00=$	2700	24097
$3\times3\times7+0=$	630	
$7^2=$	49	
	3379	23653
	410700	444851
$3\times37\times1+0=$	1110	
$1^2=$	1	
	411811	411811
	4129230000	33040963712
$3\times3710\times8+0=$	890400	
$8^2=$	64	
	4130120464	33040963712

The fourth root of a number, or algebraic quantity, may be found by first extracting the square root, and then the square root of *that* square root. Thus, $\sqrt[4]{256}=\sqrt{\sqrt{256}}=\sqrt{16}=4$.

In the same manner, the sixth root may be found by extracting the cube root, and then the square root of *that* cube root. Thus, $\sqrt[6]{x^6}=\sqrt{\sqrt[3]{x^6}}=\sqrt{x^2}=x$, &c.

61. The roots of algebraic quantities may be expressed by fractional indices.

Square root of $a^2=a=a^{\frac{2}{2}}$; the cube root of $a^6=a^2=a^{\frac{6}{3}}$.

In the same manner, the nth root of $a^m=a^{\frac{m}{n}}$, from which it appears, that the roots of quantities are found by dividing the index of the quantity by the index of the root. Hence, square root of a or $\sqrt{a}=a^{\frac{1}{2}}$; cube root of a or $\sqrt[3]{a}=a^{\frac{1}{3}}$; cube root of a^2 or $\sqrt[3]{a^2}=a^{\frac{2}{3}}$; the fifth root of a^3 or $\sqrt[5]{a^3}=a^{\frac{3}{5}}$. In like manner, the square root of $a+x$ or $\sqrt{a+x}=(a+x)^{\frac{1}{2}}$; the cube root of $(a+x)^2$ or $\sqrt[3]{(a+x)^2}=(a+x)^{\frac{2}{3}}$, &c.

62. Any quantity raised to the power, whose index is 0, is equal to unity.

For $\frac{a^m}{a^n}=a^{m-n}$, (No. 29.); now, if $n=m$, then $a^{m-n}=a^0$, and $\frac{a^m}{a^n}$ will be the same as $\frac{a^m}{a^m}$, which is $=1$, because a quantity divided by itself $=$ unity; hence $a^0=1$.

63. The powers and roots of quantities may be expressed by *negative* indices.

Thus, $\frac{a^2}{a^5}=a^{2-5}=a^{-3}$, but $\frac{a^2}{a^5}=\frac{1}{a^3}$, by reducing the fraction to its lowest terms, hence, $\frac{1}{a^3}=a^{-3}$. In the same manner, $\frac{1}{a^2}=a^{-2}$; $\frac{1}{a^5}=a^{-5}$; $\frac{1}{a^m}=a^{-m}$. Conversely, $\frac{a^5}{a^2}=$

$\frac{1}{a^{2-5}} = \frac{1}{a^{-3}}$, but $\frac{a^5}{a^2} = a^3$, hence, $a^3 = \frac{1}{a^{-3}}$. In the same manner, $a^5 = \frac{1}{a^{-5}}$, $a^m = \frac{1}{a^{-m}}$, &c.

64. From what is proved in the preceding number, it follows, that any quantity may be transferred from the numerator of a fraction into the denominator, or from the denominator into the numerator, by changing the sign of its index.

Thus, $\frac{a^3}{b^5} = a^3 \times \frac{1}{b^5} =$ (since $\frac{1}{b^5} = b^{-5}$) $a^3 b^{-5}$. In like manner, $\frac{a^2b^3c^5}{d^3e^4x^2} = a^2b^3c^5d^{-3}e^{-4}x^{-2}$, or to $\frac{1}{a^{-2}b^{-3}c^{-5}d^3e^4x^2}$.

65. A compound quantity, having a fractional index, may be expanded into an infinite series, by means of the Binomial Theorem. For it may be demonstrated, that the general formula stated in No. 50, is equally true, whether n be positive or negative, integral or fractional.

The general formula may perhaps be most conveniently expressed in the following form:

$$(a+b)^n = a^n + Pa^{n-1}b + Qa^{n-2}b^2 + Ra^{n-3}b^3 + Sa^{n-4}b^4 + \&c.$$

Or, $= a^n(1 + P\frac{b}{a} + Q\frac{b^2}{a^2} + R\frac{b^3}{a^3} + S\frac{b^4}{a^4} + \&c.)$ where in both the coefficient $P = n$.

$$Q = P.\frac{n-1}{2}.$$

$$R = Q.\frac{n-2}{3}.$$

$S = R.\frac{n-3}{4}$, &c. as will appear evident from comparing them with the coefficients of $(a+b)^n$, stated in the number already alluded to.

The first of these forms is most convenient when n is integral and positive; the second when n is fractional or negative.

The application of the Binomial Theorem has already been shown when n is a whole number; let us now endeavour to apply it when n is fractional or negative.

Ex. 1. Required the square root of $(a+b)$. That is, required to raise $(a+b)$ to the power whose index is $\frac{1}{2}$.

First, To find the coefficients.

Here $P=\frac{1}{2}$.

$$Q=\frac{1}{2}\times\frac{\frac{1}{2}-1}{2}=\frac{1}{2}\times\frac{\frac{1}{2}-\frac{2}{2}}{2}=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}=\frac{1}{2}\times-\frac{1}{4}=-\frac{1}{8}.$$

$$R=-\frac{1}{8}\times\frac{\frac{1}{2}-2}{3}=-\frac{1}{8}\times\frac{\frac{1}{2}-\frac{4}{2}}{3}=-\frac{1}{8}\times\frac{-\frac{3}{2}}{3}=-\frac{1}{8}\times-\frac{1}{2}$$
$$=\frac{1}{16}.$$

$$S=\frac{1}{16}\times\frac{\frac{1}{2}-3}{4}=\frac{1}{16}\times\frac{\frac{1}{2}-\frac{6}{2}}{4}=\frac{1}{16}\times\frac{-\frac{5}{2}}{4}=\frac{1}{16}\times-\frac{5}{8}=$$
$$-\frac{5}{128}.$$

Hence, $(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}b-\frac{1}{8}a^{-\frac{3}{2}}b^2+\frac{1}{16}a^{-\frac{5}{2}}b^3-\frac{5}{128}a^{-\frac{7}{2}}b^4+$ &c.

The negative indices of a may be rendered positive by transferring the quantity from the numerator to the denominator of the fraction, (No. 64.) We shall then have,

$$(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}}+\frac{b^3}{16a^{\frac{5}{2}}}-\frac{5b^4}{128a^{\frac{7}{2}}}+\text{&c.}$$

If the second formula be used, which, in this instance, is preferable,

$$(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}\left(1+\frac{1}{2}\frac{b}{a}-\frac{1}{8}\frac{b^2}{a^2}+\frac{1}{16}\frac{b^3}{a^3}-\frac{5}{128}\frac{b^4}{a^4}+\text{&c.}\right)=$$
$$a^{\frac{1}{2}}\left(1+\frac{b}{a}-\frac{b^2}{8a^2}+\frac{b^3}{16a^3}-\frac{5b^4}{128a^4}+\text{&c.}\right)$$

This, with every such example, is evidently an *infinite* series, since none of the factors which compose the coefficients can ever become $=0$.

Ex. 2. Required the cube root of the square of $(a^2 - x^2)$.

This result will obviously be indicated by $(a^2 - x^2)^{\frac{2}{3}}$.

The coefficients obtained, as in the preceding example, will be, 1*st*, $P = \frac{2}{3}$; 2*d*, $Q = -\frac{1}{9}$; 3*d*, $R = \frac{4}{81}$; 4*th*, $S = -\frac{7}{243}$, &c.

Hence, $(a^2 - x^2)^{\frac{2}{3}} = a^{\frac{2}{3}}\left(1 - \frac{2x^2}{3a^2} - \frac{x^4}{9a^4} - \frac{4x^6}{81a^6} - \frac{7x^8}{243a^8} - \text{\&c.}\right)$

Leaving the coefficients out of consideration, the terms of $(a^2 - x^2)^{\frac{2}{3}}$ would evidently be + and — alternately; because x^2 being a negative quantity, all its odd powers will be negative, and its even powers positive, (No. 48.) If, however, any of the coefficients of these positive terms should come out negative, then we shall have a positive quantity multiplied by a negative; and, consequently, (No. 22.) the product is negative, that is, the term will become negative. Thus, the literal part $\frac{x^4}{a^4}$ of the third term, in the above example, is naturally +, the coefficient of it, however, comes out $-\frac{1}{9}$, and hence the term is —. In short, if any coefficient, found as before, have the same sign as the term to which it belongs naturally has, that term will be positive; but if it have a different sign, the term will be negative.

Ex. 3. Expand $\frac{a}{(x-y)^{\frac{1}{3}}}$ into an infinite series.

$\frac{a}{(x-y)^{\frac{1}{3}}} = a \times (x-y)^{-\frac{1}{3}}$ (No. 64.) The coefficients are $P = -\frac{1}{3}$, $Q = \frac{2}{9}$. $R = -\frac{14}{81}$, $S = \frac{35}{243}$. Now, $(x-y)^{-\frac{1}{3}} = x^{-\frac{1}{3}}\left(1 + \frac{1}{3}\frac{y}{x} + \frac{2}{9}\frac{y^2}{x^2} + \frac{14}{81}\frac{y^3}{x^3} + \frac{35}{243}\frac{y^4}{x^4} + \text{\&c.}\right) = \frac{1}{x^{\frac{1}{3}}}\left(1 + \frac{y}{3x} + \frac{2y^2}{9x^2} + \frac{14y^3}{81x^3} + \frac{35y^4}{243x^4} + \text{\&c.}\right)$ And hence, $a \times (x-y)^{-\frac{1}{3}}$ or

$$\frac{a}{(x-y)^{\frac{1}{3}}} = \frac{a}{x^{\frac{1}{3}}}\left(1 + \frac{y}{3x} + \frac{2y^2}{9x^2} + \frac{14y^3}{81x^3} + \frac{35y^4}{243x^4} + \text{\&c.}\right)$$

From the preceding example, it appears, that the Binomial Theorem may not only be employed in extracting imperfect roots, but also in finding quotients. Thus, $\frac{a}{(x-y)^{\frac{1}{3}}}$, evidently expresses the quotient of a divided by the cube root of $(x-y)$. If, by means of this theorem, the student find the quotients of a divided by $a+x$, and of a divided by $a-x$, or, which is the same thing, expand $a(a+x)^{-1}$, and $a(a-x)^{-1}$ into infinite series, he will find the results to agree with those stated on page 68, as found by actual division.

The Binomial Theorem may be very conveniently employed for extracting imperfect roots of numbers, more particularly when the number exceeds the complete power, whose root is to be extracted by a very little.

Ex. Required the cube root of 29.

$29^{\frac{1}{3}}=(27+2)^{\frac{1}{3}}=27^{\frac{1}{3}}(1+\frac{1}{3}\times\frac{2}{27}-\frac{1}{9}\times\frac{2^2}{27^2}+\frac{5}{81}\times\frac{2^3}{27^3}-$ &c.$)=3(1+\frac{2}{81}-\frac{4}{6561}+\frac{40}{1594323}-$&c.$)=$ (rejecting the last fraction) $3(1+.0240817)=3.0722451$ nearly.

In all examples of this kind, when the second term of the binomial is small in comparison of the first, it is obvious that a few of the first terms of the expanded form will be sufficiently accurate for most purposes.

EXAMPLES.

Ex. 1. Convert $(a-x)^{\frac{1}{2}}$ into an infinite series.

Ex. 2. Convert $(a^2+x^2)^{\frac{3}{5}}$ into an infinite series.

Ex. 3. Required the quotient of a^2 divided by the cube root of the square of x^2+y^2. *That is*, required to convert $\frac{a^2}{(x^2+y^2)^{\frac{2}{3}}}$ into an infinite series.

Ex. 4. Convert $\frac{1}{\overline{1+x}|^2}$ and $\frac{1}{\overline{1-x}|^2}$, into infinite series, and then add the two series together.

Ex. 5. Required the cube root of 67, or of its equal 64+3.

SURDS.

66. The definition of a surd has already been given in No. 55. The precise values of surd quantities cannot be ascertained. They may, however, be determined to any degree of exactness less than perfect, by means of decimals or infinite series.

In this sense, surds quantities are called *irrational,* to distinguish them from those, whether integral or fractional, whose values are determinate, and which are therefore called *rational.* Thus, $a+x$, 5, are rational quantities; $(a+x)^{\frac{1}{2}}$, $\sqrt{5}$, are irrational.

Surds, when properly reduced, are subject to the common elementary rules of Arithmetic and Algebra.

ON THE REDUCTION OF SURDS.

Case I.

67. *A rational quantity may be reduced to the form of a surd, by raising it to the power corresponding to the root of the given surd, and then prefixing the radical sign with the proper index.*

Ex. 1. Reduce 5 to the form of the square root.

$5=\sqrt{5^2}=\sqrt{25}$. Ans.

Ex. 2. Reduce $\frac{3}{4}$ to the form of the cube root.

$\frac{3}{4}=\sqrt[3]{\frac{3^3}{4^3}}=\sqrt[3]{\frac{27}{64}}$. Ans.

Ex. 3. Reduce $a+x$ to the form of the square root.

$a+x=\sqrt{(a+x)^2}$ or to $\sqrt{a^2+2ax+x^2}$.

Ex. 4. Reduce $3a^{\frac{1}{3}}$ to the form of the cube root.

$3a^{\frac{1}{3}}=\sqrt[3]{\left(3a^{\frac{1}{3}}\right)^3}=\sqrt[3]{27a}$. Ans.

CASE II.

68. *Surd quantities, having different indices, may be reduced to other equivalent ones, having a common index, by reducing their indices to a common denominator.*

Ex. 1. Reduce $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ to quantities having the same radical sign.

The fractions $\frac{1}{2}$ and $\frac{1}{3}$, when reduced to a common denominator, are $\frac{3}{6}$ and $\frac{2}{6}$ respectively. Therefore, $a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}$, and $a^{\frac{1}{3}} = a^{\frac{2}{6}} = \sqrt[6]{a^2}$; hence, $\sqrt[6]{a^3}$ and $\sqrt[6]{a^2}$, are the quantities required.

Ex. 2. Reduce $2^{\frac{2}{3}}$ and $3^{\frac{1}{2}}$ to quantities having the same radical sign.

The fractions $\frac{2}{3}$ and $\frac{1}{2}$ are $= \frac{4}{6}$ and $\frac{3}{6}$. Therefore, $2^{\frac{2}{3}}$ and $3^{\frac{1}{2}}$ are $= 2^{\frac{4}{6}}$ and $3^{\frac{3}{6}}$, or to $\sqrt[6]{2^4}$ and $\sqrt[6]{3^3} =$ (by actually involving the quantities within the radical sign) $\sqrt[6]{16}$ and $\sqrt[6]{27}$, the quantities required.

Ex. 3. Reduce $3\sqrt[3]{2}$ and $2\sqrt{5}$ to surds having the same radical sign.

$3\sqrt[3]{2}$ and $2\sqrt{5} = 3 \times 2^{\frac{1}{3}}$, and $2 \times 5^{\frac{1}{2}} = 3 \times 2^{\frac{2}{6}}$, and $2 \times 5^{\frac{3}{6}} = 3\sqrt[6]{2^2}$, and $2\sqrt[6]{5^3} = 3\sqrt[6]{4}$, and $2\sqrt[6]{125}$, the quantities required.

CASE III.

69. *Surds admit of being simplified when the quantity under the radical sign can be resolved into two factors, one of which is a complete power of the kind required.*

Ex. 1. $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$.

Ex. 2. $3\sqrt{ax^2-bx^2}=3\sqrt{x^2(a-b)}=3\sqrt{x^2}\times\sqrt{a-b}=3x\sqrt{a-b}$.

Ex. 3. $\sqrt{147}=\sqrt{49\times3}=\sqrt{49}\times\sqrt{3}=7\sqrt{3}$.

Ex. 4. Reduce $\sqrt[3]{108}$ to its simplest form. $\sqrt[3]{108}=\sqrt[3]{27\times4}=\sqrt[3]{27}\times\sqrt[3]{4}=3\sqrt[3]{4}$.

The rational part of a surd quantity may be placed within the radical sign, by raising it to the power denoted by the index of the surd.

Thus, $a\sqrt{a}=\sqrt{a^2}\times\sqrt{a}=\sqrt{a^2\times a}=\sqrt{a^3}$; $3\sqrt{2}=$ in like manner to $\sqrt{18}$; and $2\sqrt{a-x}=\sqrt{4a-4x}$, &c.

CASE IV.

70. *If the quantity under the radical sign be a fraction, it may be reduced to the integral form, by multiplying the numerator and denominator of the fraction by such a quantity as will render the denominator a complete power corresponding to the index of the root.*

Ex. 1. $a\sqrt{\frac{b}{c}}=a\sqrt{\frac{bc}{c^2}}=a\sqrt{\frac{1}{c^2}\times bc}=a\times\frac{1}{c}\sqrt{bc}=\frac{a}{c}\sqrt{bc}$.

Ex. 2. $\frac{3}{4}\sqrt[3]{\frac{2}{3}}=\frac{3}{4}\sqrt[3]{\frac{2\times3\times3}{3\times3\times3}}=\frac{3}{4}\sqrt[3]{\frac{18}{27}}=\frac{3}{4}\sqrt[3]{\frac{1}{27}\times18}=\frac{3}{4}\times\frac{1}{3}\sqrt[3]{18}=\frac{1}{4}\sqrt[3]{18}$.

EXERCISES ON THE REDUCTION OF SURDS.

Ex. 1. Reduce 2 to the form of the cube root. Ans. $\sqrt[3]{8}$.

Ex. 2. Reduce $\frac{a}{b}$ to the form of the fourth root. Ans. $\sqrt{\frac{a^4}{b^4}}$.

Ex. 3. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{3}{4}}$ to surds of the same radical sign.

Ans. $\sqrt[4]{a^2}$ and $\sqrt[4]{b^3}$.

Ex. 4. Reduce $2^{\frac{2}{3}}$ and $3^{\frac{1}{2}}$ to surds of the same radical sign. Ans. $\sqrt[6]{16}$ and $\sqrt[6]{27}$.

Ex. 5. Reduce $\sqrt{a^3b^2}$ to its simplest form. Ans. $ab\sqrt{a}$.

Ex. 6. Reduce $\sqrt[m]{a^m b^{m+2}}$ to its simplest form. Ans. $ab\sqrt[m]{b^2}$.

Ex. 7. Reduce $12\sqrt{\frac{3}{8}}$ to an equivalent quantity, having the irrational part an integer. Ans. $3\sqrt{6}$.

Ex. 8. Reduce $\frac{a}{b}\sqrt{\frac{c}{d}}$ to an equivalent quantity, having the quantity within the radical sign of the integral form.

Ans. $\frac{a}{bd}\sqrt{cd}$.

ADDITION AND SUBTRACTION OF SURDS.

71. *If, when surds are reduced to their simplest form, by Case III., the irrational part be the same in both, then their sum or difference will be equal to the sum or difference of their coefficients multiplied by the common surd part.*

Ex. 1. Required the sum and difference of $\sqrt{50}$ and $\sqrt{32}$.

By Case III. $\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$.

And, $\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$.

Hence, the sum of $\sqrt{50}$ and $\sqrt{32} = 5\sqrt{2} + 4\sqrt{2} = 9\sqrt{2}$, and their difference $= 5\sqrt{2} - 4\sqrt{2} = 1\sqrt{2}$ or simply $\sqrt{2}$.

Ex. 2. Find the sum and difference of $a\sqrt[3]{\frac{1}{2}}$ and $b\sqrt[3]{\frac{1}{16}}$.

$$a\sqrt[3]{\frac{1}{2}} = a\sqrt[3]{\frac{4}{8}} = a\sqrt[3]{\frac{1}{8} \times 4} = a \times \frac{1}{2}\sqrt[3]{4} = \frac{a}{2}\sqrt[3]{4}.$$

$$b\sqrt[3]{\frac{1}{16}} = b\sqrt[3]{\frac{4}{64}} = b\sqrt[3]{\frac{1}{64} \times 4} = b \times \frac{1}{4}\sqrt[3]{4} = \frac{b}{4}\sqrt[3]{4}.$$

Hence, the sum $= \frac{a}{2}\sqrt[3]{4} + \frac{b}{4}\sqrt[3]{4} = \left(\frac{a}{2} + \frac{b}{4}\right)\sqrt[3]{4} = \frac{2a+b}{4}\sqrt[3]{4}$;

and difference $= \left(\frac{a}{2} - \frac{b}{4}\right)\sqrt[3]{4} = \frac{2a-b}{4}\sqrt[3]{4}$.

Ex. 7. Required the square of $3\sqrt[3]{4}$. Ans. $18\sqrt[3]{2}$.

Ex. 8. Required the cube of $2\sqrt{5}$. Ans. $40\sqrt{5}$.

Ex. 9. Required the square root of $4\sqrt[3]{2}$. Ans. $2\sqrt[6]{2}$.

Ex. 10. Required the cube root of $\frac{8}{27}\sqrt{7}$. Ans. $\frac{2}{3}\sqrt[6]{7}$.

74. From what has been stated regarding the management of surd quantities, we can readily reduce a fraction which has a surd quantity in its denominator, to an equivalent fraction having a rational denominator. Thus, if both numerator and denominator of the fraction $\frac{a}{\sqrt{x}}$ be multiplied by $\sqrt{x}$, we obtain the equivalent fraction $\frac{a\sqrt{x}}{\sqrt{x}\times\sqrt{x}}=\frac{a\sqrt{x}}{x}$, having a rational denominator. In the same manner, if $\frac{b}{\sqrt[3]{a+x}}$ or $\frac{b}{(a+x)^{\frac{1}{3}}}$ have its numerator and denominator both multiplied by $(a+x)^{\frac{2}{3}}$, it will become $\frac{b(a+x)^{\frac{2}{3}}}{a+x}$. In such examples, it is evident that the index of the multiplier must just be such a fraction, as that when added to the index of the given denominator, it shall produce the least possible whole number.

EQUATIONS.

75. When one algebraic quantity is connected with another by means of the sign of equality, (=), the whole expression constitutes what is called an equation. Thus, $ax+b=c$ is an equation. In equations, as applied to the solution of problems, the object is to discover the value of certain unknown quantities, from having their relation specified or given to such as are known. When the value of the unknown quantity is found in terms of those which are given, the equation is said to be solved.

76. If an equation contain only the first power of the unknown quantity, it is called a *simple* equation. Thus, $3x+a=b$ is a simple equation. If it contain the second power of the unknown quantity, or the product of two unknown quantities, it is called a *quadratic* equation. Thus, $x^2+b=c$; $xy-a=b$ are quadratic equations. If it contain the third power of the unknown quantity, it is called a *cubic* equation. Or, in general, if it contain the nth power of the unknown quantity, it is said to be an equation of n dimensions.

77. In solving an equation, the great object is to disengage the unknown quantity from the given quantities with which it is combined, so as to make it stand on one side of the equation by itself, and thus determine its value. The disengagement of the unknown quantity, according to the particular manner of its combination, may be effected on one or more of the following axioms, or self-evident principles:

Ax. 1. If, to equal quantities, equals be added, the wholes will be equal.

Ax. 2. If, from equal quantities, equals be taken, the remainders will be equal.

Ax. 3. If equal quantities be multiplied by equals, the products will be equal.

Ax. 4. If equal quantities be divided by equals, the quotients will be equal.

Ax. 5. If two quantities be equal, any like powers or roots of them will also be equal.

78. *On the solution of simple equations containing only one unknown quantity.*

Ex. 1. Let $x+5=12$. If 5 be taken from each, then (Ax. 2.) $x+5-5=12-5$; but $5-5=0$, hence $x=12-5$.

Ex. 2. Let $x-7=9$. If 7 be added to each, then (Ax. 1.) $x-7+7=9+7$; but $-7+7=0$, hence $x=9+7$.

Ex. 3. Let $3x-6=2x+2$. If 6 be added to each, then $3x-6+6=2x+2+6$. Again, if $2x$ be subtracted from each, then $3x-6+6-2x=2x+2+6-2x$; but $-6+6=0$, and $2x-2x$ also $=0$, hence $3x-2x=2+6$.

On reviewing the three preceding examples, it appears,

1st, That $x+5=12$, is identical with - $x=12-5$.
2d, - $x-7=9$, - - - - $x=9+7$.
3d, - $3x-6=2x+2$, - - $3x-2x=2+6$.

Hence it follows, that *any quantity may be transferred from one side of the equation to the other by changing its sign.*

In the first of the above examples, it is manifest that $x=7$; in the second $x=16$; and, in the third, $x=8$.

79. From the conclusion now drawn regarding the transposition of quantities, it readily follows, *1st*, That, if the same quantity stand on both sides of the equation, with the same sign, it may be left out of both. Thus, if $x+a=b+a$, then $x=b+a-a$, or $x=b$; and, *2d*, That the signs of all the terms may be changed without, in any respect, changing the value of the unknown quantity. Thus, if $x-a=b-c$, then, by transposition, $x=a+b-c$. If the signs of all the terms be changed, then we shall have $a-x=c-b$; and hence, by transposing $-x$ to the right hand side of the equation, and $c-b$ to the left, we find $a-c+b=x$, where the value of x is the same as before.

80. If the unknown quantity has a coefficient, it may be taken away by dividing both sides of the equation by it.

Ex. 1. Let $3x=12$; then $\frac{3x}{3}=\frac{12}{3}$, (Ax. 4.); but $\frac{3x}{3}=x$, and $\frac{12}{3}=4$; hence, $x=4$.

Ex. 2. Let $5x+8=3x+16$. Then, by transposition, $5x-3x=16-8$, or $2x=8$, and hence $x=4$.

Ex. 3. Let $3ax+b=cx+d$. Then $3ax-cx=d-b$, but $3ax-cx=(3a-c)x$, (No. 27.); hence, $(3a-c)x=d-b$, and, dividing both sides of the equation by $3a-c$, we have $x=\frac{d-b}{3a-c}$.

81. *An equation may be cleared of fractions by multiplying each side of the equation by the denominators of the fractions in succession, or by multiplying each side by their least common multiple.*

Before giving an example, it may be necessary to remark, that the product of a fraction, multiplied by its denominator, is equal to the numerator. Thus, let $\frac{a}{b}$ be any fraction, then $\frac{a}{b} \times b = \frac{ab}{b} = a$.

Ex. 1. Let $\frac{x}{2} = 4$; then, by multiplying both by 2, we have $x = 8$, (Ax. 3.)

Ex. 2. Let $\frac{x}{2} + \frac{x}{3} = 10$. Then, if each side be multiplied by 2, we have $x + \frac{2x}{3} = 20$. Again, let each side be multiplied by 3, and we have $3x + 2x = 60$, that is, $5x = 60$, and hence $x = \frac{60}{5} = 12$. Or, multiplying by 6, the least common multiple of 2 and 3, we have $\frac{6x}{2} + \frac{6x}{3} = 60$, that is, $3x + 2x = 60$, whence, as before, $x = 12$.

Ex. 3. Let $\frac{ax}{b} + \frac{cx}{d} = \frac{a^2+x}{3}$. Then, multiplying by b, $ax + \frac{bcx}{d} = \frac{a^2b+bx}{3}$; again, multiply by d, we have $adx+bcx = \frac{a^2bd+bdx}{3}$; lastly, multiply by 3, we have $3adx+3bcx = a^2bd+bdx$; now, by transposition, $3adx+3bcx-bdx = a^2bd$; but $3adx+3bcx-bdx = (3ad+3bc-bd)x$, (No. 27.;) hence, $(3ad+3bc-bd)x = a^2bd$, and, dividing both sides of the equation by $3ad+3bc-bd$, we have $x = \frac{a^2bd}{3ad+3bc-bd}$.

82. *If the unknown quantity, in any equation, be in the form of a surd, it may be reduced to the rational form, by transposing the terms, so that this surd quantity may stand on one side of the equation, and the remaining terms on the other, and then involving both sides to such a power as corresponds with the index of the surd.*

Ex. 1. Let $\sqrt{x}+3 = 8$. Then, by transposition, $\sqrt{x} = 8 - 3 = 5$, now, by squaring both, we have $x = 25$, (Ax. 5.)

Ex. 2. Let $\sqrt{x+2}=7$. By squaring, we have $x+2=49$, and hence $x=49-2=47$.

Ex. 3. Let $3\sqrt{a+x}-b=c$. By transposing, $3\sqrt{a+x}=b+c$. Dividing both by 3, we have $\sqrt{a+x}=\frac{b+c}{3}$; squaring both, $a+x=\frac{(b+c)^2}{9}$, and hence, by transposing, $x=\frac{(b+c)^2}{9}-a$.

EXAMPLES.

Of simple equations containing only one unknown quantity.

Required the values of x *in the following equations:*

Ex. 1. Given $x+3=10$. Ans. $x=7$.

Ex. 2. Given $x-a+c=b$. Ans. $x=a+b-c$.

Ex. 3. Given $3x+5=2x+14$. Ans. $x=9$.

Ex. 4. Given $7x-8=2x+12$. Ans. $x=4$.

Ex. 5. Given $11x-15=7x+5$. Ans. $x=5$.

Ex. 6. Given $12x+16=4x+22$. Ans. $x=\frac{3}{4}$.

Ex. 7. Given $9x+4=3x+18$. Ans. $x=2\frac{1}{3}$.

Ex. 8. Given $\frac{2x}{3}+\frac{3x}{4}-\frac{x}{2}=11$. Ans. $x=12$.

Ex. 9. Given $\frac{3x+4}{5}+2x=\frac{22-x}{5}+16$. Ans. $x=7$.

Ex. 10. Given $x-\frac{2x+1}{3}=\frac{x+3}{4}$. Ans. $x=13$.

Ex. 11. Given $\frac{7x-8}{11}+\frac{15x+8}{13}=3x-\frac{31-x}{2}$. Ans. $x=9$.

Ex. 12. Given $\frac{3a+x}{x}-5=\frac{6}{x}$. Ans. $x=\frac{3a-6}{4}$.

Ex. 13. Given $4x+\frac{1}{10}-\frac{3x-13}{16}-\frac{12+7x}{9}=7x-33-\frac{9+5x}{10}-\frac{11x-17}{8}$. Ans. $x=15$.

Ex. 14. Given $\sqrt{x}-7=2$. Ans. $x=81$.

Ex. 15. Given $3\sqrt{2x+4}=12$. Ans. $x=6$.

Ex. 16. Given $\sqrt{1+x\sqrt{x^2+12}}=1+x$. Ans. $x=2$.

On the Solution of Simple Equations containing Two or more unknown Quantities.

83. For the solution of equations containing two or more unknown quantities, as many independent equations are required as there are unknown quantities.

There are three general methods by which the solution of equations of this kind may be effected.

84. METHOD 1*st*, In the case of two unknown quantities, *multiply the terms of the first equation by the coefficient of one of the unknown quantities in the second, and the terms of the second equation by the coefficient of the same unknown quantity in the first; subtract or add these resulting equations, according as the signs of this unknown quantity in the two cases are the same or different, there will, from this process, arise an equation containing only one unknown quantity, whose value may be determined by the rules already stated.*

Ex. 1. Given $3x+2y=21$ and $2x+3y=19$, to find the values of x and y.

$(A)\ 3x+2y=21$
$(B)\ 2x+3y=19$

Multiply equation A by 2. Then $6x+4y=42$
" " B by 3. " $6x+9y=57$

Subtract the upper from the lower. Then $5y=15$

Hence, $y=\frac{15}{5}=3$.

Now, from equation A, we have $3x=21-2y$, and, consequently, $x=\frac{21-2y}{3}$. In place of y, substitute its value (3) already found, and we shall have $x=\frac{21-6}{3}=\frac{15}{3}=5$. Hence, the values of x and y are 5 and 3 respectively.

Ex. 2. Given $5x+2y=34$, and $3y-2x=13$, to find the values of x and y.

$$\begin{array}{lrcr} (A) & 5x + 2y & = & 34 \\ (B) & -2x + 3y & = & 13 \end{array}$$

Multiplying (*A*) by 2, and (*B*) by 5, we have $\left\}\begin{array}{rcr} 10x + 4y & = & 68 \\ -10x + 15y & = & 65 \end{array}\right.$

Hence, by addition, - $19y = 133$

And, therefore, $y = \frac{133}{19} = 7$.

From equation (*A*) $x = \frac{34 - 2y}{5} =$ (y being $= 7$) $\frac{34 - 14}{5} = \frac{20}{5} = 4$.

85. METHOD 2d, *Find a value, in each equation, of one of the unknown quantities in terms of the other and known quantities. These two values of the same unknown quantity being stated as equal, will form an equation containing only one unknown quantity, whose value may be determined as before.*

Ex. 1. Given $2x - 3y = 2$, and $5x + y = 56$, to find the values of x and y.

(*A*) $2x - 3y = 2$; from *A* (No. 80.) $x = \frac{2 + 3y}{2}$.

(*B*) $5x + y = 56$; - *B* - $x = \frac{56 - y}{5}$.

Hence, $\frac{2+3y}{2} = \frac{56 - y}{5}$. Or, by clearing away the fractions, (No. 81.) $10 + 15y = 112 - 2y$; $15y + 2y = 112 - 10$, that is, $17y = 102$, and, therefore, $y = \frac{102}{17} = 6$.

Now, $x = \frac{2 + 3y}{2} = \frac{2+18}{2} = \frac{20}{2} = 10$.

Ex. 2. Given $\frac{4x - 3y}{2} - \frac{2x - y}{3} = \frac{x + y}{3} - 1$, and $\frac{3x+2y}{4} + \frac{3y - 8}{2} = \frac{5x - 4y}{2} + 3\frac{1}{6}$, to find the values of x and y.

(*A*) $\frac{4x-3y}{2} - \frac{2x-y}{3} = \frac{x+y}{3} - 1$. From *A*, $x = \frac{3y - 2}{2}$.

(*B*) $\frac{3x+2y}{4} + \frac{3y - 8}{2} = \frac{5x - 4y}{2} + 3\frac{1}{4}$. - *B*, $x = \frac{16y - 29}{7}$.

Hence, $\frac{3y-2}{2}=\frac{16y-29}{7}$, from which we have $y=4$. Now, $x=\frac{3y-2}{2}=\frac{12-2}{2}=\frac{10}{2}=5$.

86. **Method 3d,** *In either equation, find a value of one of the unknown quantities, in terms of the other and known quantities; substitute this value for the unknown quantity in the second equation, there will thence arise an equation containing only one unknown quantity, the value of which may be determined as before.*

Ex. 1. Given $5x+3y=50$, and $3x-2y=11$, to find the values of x and y.

$$\begin{aligned}(A)\ 5x+3y&=50.\\(B)\ 3x-2y&=11.\end{aligned}$$

From A, $x=\frac{50-3y}{5}$. Now, by substituting this value for x in B, we have, $3\times\frac{50-3y}{5}-2y=11$, or, by multiplication, $150-9y-10y=55$, whence we have $y=5$; hence $x=\frac{50-3y}{5}=\frac{50-15}{5}=\frac{35}{5}=7$.

Ex. 2. Given $3ax-2ay=b$, and $2ax+ay=c$, to find the values of x and y.

$$\begin{aligned}(A)\ 3ax-2ay&=b.\\(B)\ 2ax+\ ay&=c.\end{aligned}$$

From A, $x=\frac{b+2ay}{3a}$. Now, by substituting this value for x in B, we have $2a\times\frac{b+2ay}{3a}+ay=c$; or, by multiplication, $2ab+4a^2y+3a^2y=3ac$, that is, $2ab+7a^2y=3ac$; whence $y=\frac{3ac-2ab}{7a^2}=\frac{3c-2b}{7a}$. Hence, $x=\frac{b+2ay}{3a}=$ (by substituting the value of y) $\dfrac{b+2a\times\frac{3c-2b}{7a}}{3a}=\dfrac{b+\frac{6c-4b}{7}}{3a}=$ $\frac{7b+6c-4b}{21a}=\frac{3b+6c}{21a}=\frac{b+2c}{7a}$.

87. The application of the preceding methods will enable us to solve equations containing *three* or *more* unknown quantities, as will readily appear from an example.

Ex. Given $x+2y+3z=20$, $2x+3y-z=22$, and $3x+y+2z=26$, to find the values of x, y and z.

$$\begin{array}{rr}
(A) & x+2y+3z=20 \\
(B) & 2x+3y-\ \ z=22 \\
(C) & 3x+\ \ y+2z=26 \\ \hline
\end{array}$$

$$\begin{array}{lrl}
\text{Multiply } A \text{ by } 2\text{; then,} & 2x+4y+6z=40 & (D) \\
\text{Subtract } B, \quad - & 2x+3y-\ \ z=22 & \\ \hline
 & y+7z=18 & (D) \\
\text{Multiply } A \text{ by } 3\text{; then,} & 3x+6y+9z=60 & \\
\text{Subtract } C, \quad - & 3x+\ \ y+2z=26 & \\ \hline
 & 5y+7z=34 & (E).
\end{array}$$

Hence the given equations are reduced to the two.

$$\begin{array}{rl}
y+\ \ 7z=18 & (D) \\
5y+\ \ 7z=34 & (E) \\ \hline
\end{array}$$

$$\begin{array}{lr}
\text{Multiply } D \text{ by } 5\text{; then,} & 5y+35z=90 \\
\text{Subtract } E, \quad - & 5y+\ \ 7z=34 \\ \hline
 & 28z=56
\end{array}$$

$$\text{Hence,} \quad - \quad z=\frac{56}{28}=2.$$

Now, from equation D, $y=18-7z=$ (by substituting 2, the value of z) $18-14=4$. Again, from equation A, $x=20-2y-3z=$ (by substitution) $20-8-6=20-14=6$. Hence the values of x, y, and z, are 6, 4 and 2 respectively.

In a similar manner may any equations be solved, whatever be the number of unknown quantities they contain, provided there be always as many independent equations, as there are unknown quantities.

EXAMPLES.

Of Equations containing two or more unknown Quantities.

Required the values of x *and* y *in the following equations:*

Ex. 1. Given $5x+y=23$, and $3x+2y=18$.
Ans. $x=4$ and $y=3$.

Ex. 2. Given $3x+2y=54$, and $4x-3y=21$.
Ans. $x=12$, $y=9$.

Ex. 3. Given $9x+2y=7$, and $3x+4y=4$.

Ans. $x=\frac{2}{3}$, $y=\frac{1}{2}$.

Ex. 4. Given $\frac{x+2y}{5}=\frac{x}{2}$, and $\frac{3x+2y}{9}=y-\frac{x}{4}$.

Ans. $x=8$, $y=6$.

Ex. 5. Given $\frac{2x-y}{8}+\frac{x-y}{4}=\frac{y}{2}-\frac{14-y}{3}$, and $\frac{3x+y}{9}+\frac{x+y-4}{18}=\frac{x}{5}+\frac{y}{4}+1$.

Ans. $x=10$, $y=8$.

Ex. 6. Given $\frac{a}{b+y}=\frac{b}{3a+x}$, and $ax+2by=c$.

Ans. $x=\frac{2b^2+6a^2+c}{3a}$, $y=\frac{3a^2-b^2+c}{3b}$.

Required the values of x, y, *and* z, *in the following equations:*

Ex. 7. Given $3x+2y+z=25$, $5x-y+3z=27$, and $2x-2y+5z=12$.

Ans. $x=5$, $y=4$, and $z=2$.

Ex. 8. Given $x+y+z=2y+5$, $2x-3y+2z=x-y-2$, and $3x+2y-2z=40+z+7$.

Ans. $x=12$, $y=10$, and $z=3$.

On the application of Simple Equations to the Solution of Problems.

88. Having represented the unknown quantity, or that whose value is to be determined, by one of the final letters of the alphabet, translate the conditions of the given problem into the language of algebra; there will thence arise an equation, from which the value of the unknown quantity may be readily determined.

Ex. 1. Divide L. 400 among three persons, A, B, and C, so that A may have twice as much as B, and B three times as much as C.

Let $x=$ C's share; then, since B is to have three times as much, his share will be $=3x$; again, because A is to have twice as much as B, and B's share is $=3x$, therefore A's $=6x$. Now, the sum of their shares is $=400$ L.

Hence $x+3x+6x=400$.

That is, - $10x=400$.

And $x=\frac{400}{10}=40=$ C's share.

Now, $3x = 3 \times 40 = 120 =$ B's share, and $6x = 6 \times 40 = 240 =$ A's. Hence, 240, 120, and 40, are the shares of A, B, and C respectively.

Ex. 2. What number is that whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 14.

Let $x =$ number required.

Then $\frac{x}{3} =$ its third part, and $\frac{x}{4} =$ its fourth part.

And $\frac{x}{3} - \frac{x}{4} = 14$. (By Quest.)

Multiply by 3; then, $x - \frac{3x}{4} = 42$.

" 4, " $4x - 3x = 168$.

That is, $x = 168$, the number required.

Ex. 3. A post is $\frac{1}{3}$d in the earth, $\frac{2}{5}$ths in the water, and 8 feet above the water, what is its whole length?

Let $x =$ length of the post.

Then $\frac{x}{3} =$ part in the earth.

And $\frac{2x}{5} =$ part in the water.

$8 =$ part of it above the water.

Hence, $x = \frac{x}{3} + \frac{2x}{5} + 8$.

Multiply by 3, then, $3x = x + \frac{6x}{5} + 24$.

" 5, " $15x = 5x + 6x + 120$.

$15x - 5x - 6x = 120$.

That is, $4x = 120$.

And hence $x = \frac{120}{4} = 30$ feet the length of the post.

Ex. 4. Divide the number 42 into two such, that the greater being divided by 5, and the less by 6, the quotients may together amount to 8.

Let $x =$ greater part.

Then $42 - x =$ less "

And $\frac{x}{5} + \frac{42 - x}{6} = 8$. (By Quest.)

$$x + \frac{210 - 5x}{6} = 40.$$

$$6x + 210 - 5x = 240.$$

$x = 240 - 210 = 30$, the greater part.

Hence, $42 - 30 = 12$, the less -

Ex. 5. A composition of copper and tin, containing 100 cubic inches, weighed 505 ounces, how many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and a cubic inch of tin, $4\frac{1}{4}$ ounces?

Let $x =$ number of cubic inches of *copper*.

Then, $100 - x =$ - - *tin*.

$x \times 5\frac{1}{4} = x \times \frac{21}{4} = \frac{21x}{4} =$ weight of the copper.

And $(100 - x)\,4\frac{1}{4} = (100 - x) \times \frac{17}{4} = \frac{1700 - 17x}{4} =$ weight of the tin.

Hence, $\frac{21x}{4} + \frac{1700 - 17x}{4} = 505$. (By the Quest.)

$$21x + 1700 - 17x = 2020.$$

$$4x = 2020 - 1700 = 320.$$

$x = \frac{320}{4} = 80$, the cubic inches of copper.

And $100 - 80 = 20$, - - tin.

Hence, $80 \times 5\frac{1}{4} = 420$, the ounces of copper.

And $20 \times 4\frac{1}{4} = 85$, - - tin.

Ex. 6. A merchant has two kinds of tea, one worth 6s. per lb., and the other 7s. 6d. per lb. How many lbs. of each must he take to form a chest of 65 lbs., which shall be worth L. 21?

Let $x =$ lbs. at 6s.

Then, $65 - x =$ - 7s. 6d.

$x \times 6 = 6x =$ value of the 6s. kind.

And $(65 - x)\,7\frac{1}{2} = (65 - x)\,\frac{15}{2} = \frac{975 - 15x}{2} =$ value of the 7s. 6d. kind.

Now, the value of the whole $=$ L. 21 $=$ 420s.

Hence, $6x + \frac{975 - 15x}{2} = 420.$

$$12x + 975 - 15x = 840.$$

$$-3x = 840 - 975 = -135.$$

Or, (No. 79.) $3x = 135$.

$$\text{And } x = \frac{135}{3} = 45 = \text{lbs. at 6s.}$$

And hence, $65 - 45 = 20 =$ lbs. at 7s. 6d.

Ex. 7. There are two numbers such, that $\frac{1}{2}$ the first, *plus* $\frac{1}{3}$ of the second, is equal to the second; and if, from $\frac{3}{8}$ of the first, there be taken $\frac{1}{6}$ of the second, the remainder is equal to 8. What are the numbers?

Let $x =$ first, and $y =$ second.

$$\left.\begin{aligned}\text{Then, } \frac{x}{2} + \frac{y}{3} &= y.\\ \text{And, } \frac{3x}{8} - \frac{y}{6} &= 8.\end{aligned}\right\} \text{(By the Quest.)}$$

From the first equation $x = \frac{4y}{3}$, and from the second, $x = \frac{192 + 4y}{9}$.

$$\text{Hence, } \frac{4y}{3} = \frac{192 + 4y}{9}.$$

$$12y = 192 + 4y.$$

$$8y = 192$$

$$y = \frac{192}{8} = 24, \text{ the second.}$$

$$\text{And, hence, } x = \frac{4y}{3} = \frac{96}{3} = 32, \text{ the first.}$$

Ex. 8. There is a certain fraction such, that if 1 be taken from its numerator, it will become $\frac{1}{3}$; but if taken from the denominator, it will become $\frac{1}{2}$. What is the fraction?

Let $x =$ its numerator, and $y =$ its denominator.

$$\text{Then, } \frac{x}{y} = \text{fraction sought.}$$

$$\left.\begin{aligned}\text{Now, } \frac{x-1}{y} &= \tfrac{1}{3}.\\ \text{And, } \frac{x}{y-1} &= \tfrac{1}{2}.\end{aligned}\right\} \text{(By the Quest.)}$$

From the first equation $x = \frac{y}{3} + 1$, and from the second, $x = \frac{y-1}{2}$.

Hence, $\frac{y-1}{2} = \frac{y}{3} + 1.$

$$y-1 = \frac{2y}{3} + 2.$$

$$3y-3 = 2y + 6.$$
$$3y-2y = 6 + 3.$$

That is, $y = 9$ the denominator.

And, hence, $x = \frac{y-1}{2} = \frac{9-1}{2} = \frac{8}{2} = 4$, the numerator. The fraction sought is therefore $\frac{4}{9}$.

EXERCISES.

Ex. 1. It is required to divide the number 24 into two such parts, that twice the greater may exceed three times the less by 3. Ans. 15 and 9.

Ex. 2. What number is that, from which, if 9 be subtracted, $\frac{3}{4}$ths of the remainder shall be 27. Ans. 45.

Ex. 3. A person, at the time he was married, was three times as old as his wife, but after they had lived together 20 years, he was only twice as old. What were their ages on their wedding-day?

Ans. Bridegroom's 60, and Bride's 20.

Ex. 4. A cistern is filled in 20 minutes by 3 pipes; one of which conveys 10 gallons more, and the other 5 gallons less than the third per minute. The cistern holds 820 gallons. How much flows through each pipe per minute?

Ans. 22, 7, and 12 gallons.

Ex. 5. If a cistern of 820 gallons be filled in 20 minutes by two spouts running *successively*, the one running 45 gallons, and the other 29 gallons per minute. How many minutes does each spout run?

Ans. 45 gallon-spout runs 15 minutes.
29 gallon-spout runs 5 minutes.

Ex. 6. A besieged garrison had such a quantity of bread, as would, if distributed to each at 10 ounces a-day, last 6 weeks; but having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces per day for 8 weeks. Required the number of men at first in the garrison. Ans. 3200.

Ex. 7. A gamester lost in the first game $\frac{1}{6}$th part, and in the second $\frac{1}{10}$th part of the money he had about him, but won in the third game $\frac{1}{3}$d part of it back. He counted his money, and found that he had won L. 3. How much had he about him at first? Ans. L. 45.

Ex. 8. A wine merchant has two kinds of wine, the one worth 3s. per quart, the other 1s. 8d. He wishes to mix both wines together in such quantities, that he may have 50 quarts worth 2s. 6d. per quart. How many quarts of each sort must he take?

Ans. $31\frac{1}{4}$ quarts of the best, and $18\frac{3}{4}$ of the other.

Ex. 9. As A and B were going to school, A first shot an arrow in the direction in which they were going, which B took up and shot forward, and so on alternately, till the arrow had passed exactly from one mile-stone to another; when it appeared that A had shot the arrow 8 times, and B 7 times. Some time afterwards, A and B were on the opposite banks of a river, the breadth of which they wished to ascertain. A first shot the arrow across the river, and it flew 13 yards beyond the bank on which B stood. B then took it up, and from the place where it had fallen, shot it back across the river; it now fell $9\frac{2}{7}$ yards beyond the bank on which A stood. Required the breadth of the river.

Ans. 100 yards.

Ex. 10. There is a certain fraction such, that if its numerator be increased by 1, and its denominator by 3, it will become $\frac{3}{4}$; but if its numerator be diminished by 3, and its denominator increased by 1, it will become $\frac{1}{2}$. What is the fraction. Ans. $\frac{8}{9}$.

Ex. 11. There is a certain number, consisting of two places of figures, which is equal to seven times the sum of its digits; and if, from the number itself, you subtract 18, the digits will be inverted. What is the number?

Ans. 42.

Ex. 12. It is required to find three numbers such, that the first, with $\frac{1}{3}$ the sum of the second and third, shall be equal to 60: the second, *minus* half the difference of the first and third, shall be equal to 28; and $\frac{1}{4}$ of the sum of the three numbers equal to the third *minus* 1.

Ans. 40, 36, and 24.

QUADRATIC EQUATIONS.

89. The definition of a quadratic equation has been already stated in No. 76.

Quadratic equations are of two kinds, *Pure* and *Adfected*. A pure quadratic is that which contains only the square of the unknown quantity. Thus, $x^2 = 49$; $x^2 + 7 = 43$, &c. are pure quadratics. An adfected quadratic is that which contains both the first and second power of the unknown quantity. Thus, $x^2 + ax = b$; $x^2 - 6x = 16$, &c. are adfected quadratics.

On the Solution of Pure Quadratic Equations.

90. For the solution of pure quadratics, the following rule may be given: *Transpose the terms of the equation in such a manner, that those which contain* x^2 *may stand on one side of the equation, and the known quantities on the other; divide by the coefficient of* x^2*, if it have one, then extract the square root of both sides of the equation, and the result will give the value of* x.

Ex. 1. Given $x^2 + 11 = 20$, to find the value of x.

$$x^2 + 11 = 20.$$

$$x^2 = 20 - 11 = 9. \quad \text{(Transp.)}$$

And hence, $x = \pm\sqrt{9} = \pm 3$.[*]

Ex. 2. Given $5x^2 + 7 = 3x^2 + 39$, to find the value of x.

$$5x^2 + 7 = 3x^2 + 39.$$

$$5x^2 - 3x^2 = 39 - 7. \quad \text{(Transp.)}$$

That is, $2x^2 = 32$.

$$x^2 = \frac{32}{2} = 16.$$

And hence, $x = \pm\sqrt{16} = \pm 4$.

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[*] The signs ± prefixed, indicate that the value of x is either $+3$ or -3. That this is the case, in every such instance, is manifest from the consideration, that the square root of a positive quantity is either + or −. Thus, $+3 \times +3 = +9$, and -3×-3, also $= +9$. Hence it follows, that the unknown quantity in every quadratic equation will have two values.

Ex. 3. Given $ax^2+b=c$, to find the value of x.

$$ax^2+b=c.$$
$$ax^2=c-b.$$
$$x^2=\frac{c-b}{a}.$$

And hence, $x=\sqrt{\frac{c-b}{a}}$.

When the value of the square of the unknown quantity is not a complete power, the value of the simple quantity can only be obtained by approximation.

Ex. 4. Given $\frac{5x^2}{3}-7=x^2+25$, to find the value of x.

$$\frac{5x^2}{3}-7=x^2+25.$$
$$5x^2-21=3x^2+75.$$
$$5x^2-3x^2=75+21=96.$$
$$2x^2=96.$$
$$x^2=48.$$

Hence, $x=\sqrt{48}=\pm 6.928$.

EXERCISES.

Ex. 1. Given $7x^2-12=5x^2+20$ to find x. Ans. $x=4$.

Ex. 2. Given $\frac{2x^2}{3}+\frac{x^2}{4}=x^2-3$ to find x. Ans. $x=6$.

Ex. 3. Given $\frac{ax^2}{b}+\frac{cx^2}{d}=a^2-x^2$, to find x.

Ans. $x=\sqrt{\frac{a^2bd}{ad+bc+bd}}$.

Ex. 4. Given $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$, and $\frac{2}{xy}=\frac{1}{9}$, to find x and y.

Ans. $\begin{cases}x=6 \text{ or } 3.\\ y=3 \text{ or } 6.\end{cases}$

On the Solution of Adfected Quadratic Equations.

91. Since the square of a binomial is equal to the squares of its two terms, together with twice their product, (p. 82,) it follows, that if the square of either of the terms, and

twice the product of the two, be given, the other, or its square, may be found. For it is manifest, that the coefficient, or quantity which multiplies the given term in twice the product, is equal to twice the other term; and hence the *half* of this coefficient will be the other term itself. Thus, if x^2+8x be the square of the first term, and twice the product of the two terms of a binomial, then the other term will be equal to $\frac{8}{2}=4$, and hence its square $=16$. The complete square, therefore, of the binomial $x+4$ is $= x^2+8x+16$. In the same manner may the square be completed in every instance.

This process of adding the third term to the two first terms of the square of a binomial is very appropriately called *completing* the *square*, and is of the most essential service in the solution of affected quadratics.

92. From what is stated in the preceding number, we are furnished with the following rule for the solution of affected quadratics.

Arrange the terms, containing the unknown quantity, on one side of the equation, beginning with that involving the highest power, and transpose all the known quantities to the other side. Then, if that term containing the highest power of the unknown quantity, has any coefficient, either positive or negative, let all the terms be divided by it. If now the square of half the coefficient of the second term be added to both sides of the equation, that side which involves the unknown quantity will become a complete square. Lastly, extract the square root of both sides of the equation, and a simple equation will be obtained, from which the values of the unknown quantity may be easily determined.

EXAMPLES.

Ex. 1. Given $x^2+8x=48$, to find the values of x.

$$x^2+8x=48.$$

Comp. square, $x^2+8x+16=48+16=64$.

Ext. root, $x+4=\pm\sqrt{64}=\pm 8$.

Hence by transp. $x=\pm 8-4=4$ or -12.

Ex. 2. Given $x^2 - 6x + 7 = 23$, to find the values of x.

$$x^2 - 6x + 7 = 23.$$

$$\text{Transp. } x^2 - 6x = 23 - 7 = 16.$$

$$\text{Comp. square, } x^2 - 6x + 9 = 16 + 9 = 25.$$

$$\text{Ext. root, } x - 3 = \pm\sqrt{25} = \pm 5.$$

$$\text{Transp. } x = \pm 5 + 3 = 8 \text{ or } -2.$$

Ex. 3. Given $x^2 + ax = b$, to find the values of x.

$$x^2 + ax = b.$$

$$\text{Comp, square, } x^2 + ax + \frac{a^2}{4} = b + \frac{a^2}{4} = \frac{4b + a^2}{4}.$$

$$\text{Ext. root, } x + \frac{a}{2} = \pm\sqrt{\frac{4b + a^2}{4}} = \pm\frac{\sqrt{4b + a^2}}{2}.$$

$$\text{Transp. } x = \frac{\pm\sqrt{4b + a^2}}{2} - \frac{a}{2} = \frac{\pm\sqrt{4b + a^2} - a}{2}.$$

Ex. 4. Given $3x^2 + 12x - 15 = 165$, to find the values of x.

$$3x^2 + 12x - 15 = 165.$$

$$\text{Transp. } 3x^2 + 12x = 165 + 15 = 180.$$

$$\text{Divide by } 3,\ x^2 + 4x = 60.$$

$$\text{Comp. square, } x^2 + 4x + 4 = 64.$$

$$\text{Ext. root, } x + 2 = \pm\sqrt{64} = \pm 8.$$

$$\text{Transp. } x = \pm 8 - 2 = 6 \text{ or } -10.$$

Ex. 5. Given $\frac{2x^2}{3} - \frac{3x}{2} + 9\frac{1}{2} = \frac{3x^2}{4} - 2x + 7\frac{1}{4}$, to find the values of x.

$$\frac{2x^2}{3} - \frac{3x}{2} + 9\tfrac{1}{2} = \frac{3x^2}{4} - 2x + 7\tfrac{1}{4}.$$

The equation must first be cleared of fractions, which may be done by multiplying by 12, the least common multiple of the denominators 3, 2, and 4, we shall then have $8x^2 - 18x + 114 = 9x^2 - 24x + 87$.

$$\text{Transp. } 8x^2 - 9x^2 - 18x + 24x = 87 - 114.$$

That is, $-x^2 + 6x = -27$; and, by changing the signs of all the terms, $x^2 - 6x = 27$.

$$\text{Comp. square, } x^2 - 6x + 9 = 27 + 9 = 36.$$

$$\text{Ext. root, } x - 3 = \pm\sqrt{36} = \pm 6.$$

$$\text{Transp. } x = \pm 6 + 3 = 9 \text{ or } -3.$$

Ex. 6. Given $3x^2+2x=161$, to find the values of x.

$$3x^2+2x=161.$$

Divide by 3, $x^2+\frac{2x}{3}=\frac{161}{3}$. The coefficient of x in this example is $\frac{2}{3}$, the half of which is $\frac{1}{3}$, and, therefore, its square $\frac{1}{9}$. Hence, $x^2+\frac{2x}{3}+\frac{1}{9}=\frac{161}{3}+\frac{1}{9}=\frac{483}{9}+\frac{1}{9}=\frac{484}{9}$.

Ext. root, $x+\frac{1}{3}=\pm\sqrt{\frac{484}{9}}=\pm\frac{22}{3}$.

Transp. $x=\pm\frac{22}{3}-\frac{1}{3}=\frac{21}{3}$, or $-\frac{23}{3}=7$, or $-\frac{23}{3}$.

Ex. 7. Given $x^2-12x+45=10$, to find the values of x.

$$x^2-12x+45=10.$$

Transp. $x^2-12x=10-45=-35$.

Comp. square, $x^2-12x+36=36-35=1$.

Ext. root, $x-6=\pm\sqrt{1}=\pm1$.

And hence, $x=\pm1+6=7$ or 5.

In this example, both values of x are positive. This will always be the case when the coefficient of x is greater than x itself, and is, at the same time, negative.

93. All equations containing two different powers of x, may be solved in the same manner, provided the index of x in the one term be the double of that in the other.

Ex. 8. Given $x^4-3x^2=4$, to find the values of x.

$$x^4-3x^2=4.$$

Comp. square, $x^4-3x^2+\frac{9}{4}=4+\frac{9}{4}=\frac{25}{4}$.

Ext. root, $x^2-\frac{3}{2}=\sqrt{\frac{25}{4}}=\frac{5}{2}$, (rejecting the neg. value.)

Transp. $x^2=\frac{5}{2}+\frac{3}{2}=\frac{8}{2}=4$.

Ext. root, $x=\sqrt{4}=2$.

Ex. 9. Given $x^m + bx^{\frac{m}{2}} = a$, to find the value of x.

$$x^m + bx^{\frac{m}{2}} = a.$$

Comp. square, $x^m + bx^{\frac{m}{2}} + \frac{b^2}{4} = a + \frac{b^2}{4} = \frac{4a+b^2}{4}$.

Ext. root, $x^{\frac{m}{2}} + \frac{b}{2} = \sqrt{\frac{4a+b^2}{4}} = \frac{\sqrt{4a+b^2}}{2}$.

Transp. $x^{\frac{m}{2}} = \frac{\sqrt{4a+b^2}}{2} - \frac{b}{2} = \frac{\sqrt{4a+b^2}-b}{2}$.

Ext. the mth root, $x^{\frac{1}{2}} = \left(\frac{\sqrt{4a+b^2}-b}{2}\right)^{\frac{1}{m}}$.

Squaring both, $x = \left(\frac{\sqrt{4a+b^2}-b}{2}\right)^{\frac{2}{m}}$.

Ex. 10. Given $x - 2\sqrt{x} = 3$, to find the value of x.

$$x - 2\sqrt{x} = 3.$$

Comp. square, $x - 2\sqrt{x} + 1 = 3 + 1 = 4$.

Ext. $x^{\frac{1}{2}} - 1 = \sqrt{4} = 2$.

Transp. $x^{\frac{1}{2}} = 2 + 1 = 3$.
Squaring $x = 9$.

EXERCISES.

Required the positive values of x *in the following equations:*

Ex. 1. Given $x^2 + 10x = 24$. Ans. $x = 2$.
Ex. 2. Given $x^2 - 4x + 7 = 12$. Ans. $x = 5$.
Ex. 3. Given $2x^2 + 6x - 11 = 129$. Ans. $x = 7$.
Ex. 4. Given $5x^2 + 3x = 2x^2 + 11x + 128$. Ans. $x = 8$.
Ex. 5. Given $\frac{2x^2}{3} + 5x = x^2 + x + 12$. Ans. $x = 6$.
Ex. 6. Given $\frac{5}{x-2} + \frac{x}{5} = 2\frac{5}{8}$. Ans. $x = 10$.
Ex. 7. Given $\frac{x}{x+1} + \frac{x+1}{x} = 2\frac{1}{12}$. Ans. $x = 3$.
Ex. 8. Given $\frac{3x}{x+2} - \frac{x-1}{6} = x - 9$. Ans. $x = 10$.

Ex. 9. Given $x^2 - 21x + 183 = 75$. Ans. $x = 9$ or 12.

Ex. 10. Given $4x^{\frac{1}{3}} + x^{\frac{1}{6}} = 39$. Ans. $x = 729$, or $\left(\frac{13}{4}\right)^6$.

Ex. 11. Given $ax^{\frac{2}{3}} + x^{\frac{1}{3}} = b$. Ans. $x = \frac{\sqrt{4ab+1} - 1}{2a}$.

Ex. 12. Given $x + 2 + \sqrt{x+2} = 12$. Ans. $x = 7$.

On the application of Quadratic Equations to the Solution of Problems.

94. Let the conditions of the given problem be first translated into the language of Algebra, as in Simple Equations; there will thence arise an equation from which the value of the unknown quantity may be easily determined, by the rule stated in No. 92.

In some instances, both values of the unknown quantity will answer the conditions of the proposed problem, in others only one, as will readily appear from the nature of the problem itself.

Ex. 1. It is required to divide the number 18 into two such parts that their product may be 56.

Let $x =$ the one part.

Then $18 - x =$ the other.

And $(18 - x)x = 18x - x^2 =$ product of the two parts.

Hence $18x - x^2 = 56$, (by Quest.;) or changing the signs, $x^2 - 18x = -56$.

Comp. square, $x^2 - 18x + 81 = -56 + 81 = 25$.

Ext. root, $x - 9 = \pm\sqrt{25} = \pm 5$.

Transp. $x = \pm 5 + 9 = 14$ or 4, where it appears that the two values of x are just the two parts into which the given number was required to be divided.

Ex. 2. The difference of two numbers is 5, and $\frac{2}{3}$ of their product is equal to four times the less number *plus* 20. Required the numbers,

Let $x =$ less number.

Then $x + 5 =$ greater.

And $\frac{2}{3}$ of $(x + 5)x = \frac{2}{3}$ of $(x^2 + 5x) = \frac{2x^2 + 10x}{3} = \frac{2}{3}$ of their product. Hence, $\frac{2x^2 + 10x}{3} = 4x + 20$, (by Quest.)

Multiply by 3, $2x^2 + 10x = 12x + 60$.

Transp. $2x^2 + 10x - 12x = 60$.

That is, $2x^2 - 2x = 60$.

Divide by 2, $x^2 - x = 30$.

Comp. square, $x^2 - x + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4}$.

Ext. root, $x - \frac{1}{2} = \sqrt{\frac{121}{4}} = \frac{11}{2}$.

Transp. $x = \frac{11}{2} + \frac{1}{2} = \frac{12}{2} = 6$ the less.

Hence, $x + 5 = 6 + 5 = 11$ the greater.

Ex. 3. The product of two numbers is 60, and if 5 be subtracted from the greater, and 2 added to the less, the product of the sum and remainder will also be 60. Required the numbers.

Let $x =$ the greater.

Then $\frac{60}{x} =$ the less.

And $(x - 5) \times \left(\frac{60}{x} + 2\right) = 60$.

That is, $\frac{60x - 300}{x} + 2x - 10 = 60$.

Multiply by x, then $60x - 300 + 2x^2 - 10x = 60x$.

And, by cancelling the $60x$, since it appears on both sides with the same sign, we have,

$$-300 + 2x^2 - 10x = 0.$$

Transp. $2x^2 - 10x = 300$.

Dividing $x^2 - 5x = 150$.

Comp. square, $x^2 - 5x + \frac{25}{4} = 150 + \frac{25}{4} = \frac{625}{4}$.

Ext. root, $x - \frac{5}{2} = \sqrt{\frac{625}{4}} = \frac{25}{2}$.

Transp. $x = \frac{25}{2} + \frac{5}{2} = \frac{30}{2} = 15$, the greater.

Hence, $\frac{60}{x} = \frac{60}{15} = 4$, the less number.

Ex. 4. A mirror, 24 inches by 18, is to be so framed, that the area of the front of the frame may be $\frac{2}{3}$ of that of the mirror. Required its breadth.

Let $x =$ breadth.

Then $24 + 2x =$ length of the frame.

And $(24 + 2x)x = 24x + 2x^2 =$ area of one of the side pieces. Now, since there are two of them, $(24x + 2x^2)2 = 48x + 4x^2 =$ area of the two side pieces, $18x =$ area of one of the end pieces, and $18x \times 2$, $36x =$ area of the two end pieces.

Hence, $48x + 4x^2 + 36x = 4x^2 + 84x =$ area of the whole frame.

$$\text{Now, } 4x^2 + 84x = \frac{2}{3} \text{ of } 432, \text{ (the area of the mirror.)}$$

$$\text{That is, } 4x^2 + 84x = 288.$$

$$\text{Or, } x^2 + 21x = 72.$$

$$\text{Comp. sq., } x^2 + 21x + \frac{441}{4} = 72 + \frac{441}{4} = \frac{729}{4}.$$

$$\text{Ext. root, } x + \frac{21}{2} = \sqrt{\frac{729}{4}} = \frac{27}{2}.$$

$$\text{Hence, } x = \frac{27}{2} - \frac{21}{2} = \frac{6}{2} = 3 \text{ inches, the breadth of the frame.}$$

Ex. 5. A and B set off at the same time to a place at the distance of 300 miles. A travels at the rate of one mile an hour faster than B, and arrives at his journey's end ten hours before him. What were their respective rates of travelling?

Let $x =$ A's rate of travelling per hour.

Then, $x - 1 =$ B's - - - -

$\frac{300}{x} =$ time taken on the journey by A.

$\frac{300}{x-1}$ - - - - - B.

$$\text{Hence, } \frac{300}{x} + 10 = \frac{300}{x-1}, \text{ (by the Quest.)}$$

$$300 + 10x = \frac{300x}{x-1}.$$

$$290x + 10x^2 - 300 = 300x.$$

$$\text{Transp. } 10x^2 + 290x - 300x = 300.$$

$$10x^2 - 10x = 300.$$

$$x^2 - x = 30.$$

$$\text{Comp. square, } x^2 - x + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4}.$$

$$\text{Ext. root, } x - \frac{1}{2} = \sqrt{\frac{121}{4}} = \frac{11}{2}.$$

$$x = \frac{11}{2} + \frac{1}{2} = \frac{12}{2} = 6 \text{ miles, A's rate.}$$

Hence, $x - 1 = 6 - 1 = 5$, B's rate of travelling.

EXERCISES.

Ex. 1. What two numbers are those whose sum is 17; and whose difference, multiplied by the less, is equal to 30?

Ans. 11 and 6.

Ex. 2. A and B distribute each 20s. among a certain number of beggars. A relieves 20 persons more than B; but B gives 2d. to each more than A does. How many persons were relieved by A and B respectively?

Ans. 60 by A, and 40 by B.

Ex. 3. Two persons, A and B, were despatched at the same time to a place at the distance of 90 miles; the former of whom travelled one mile an hour faster than the other, and arrived at the journey's end one hour sooner. At what rate did each person travel?

Ans. A 10, and B 9 miles per hour.

Ex. 4. What number is that whose square is greater than its simple power by 110? Ans. 11.

Ex. 5. A gentleman ordered 10 guineas to be distributed among a certain number of poor people; but, before the distribution took place, there came in six claimants more, by which means, each of the former received 4s. less than he or she would otherwise have done. How many were there at first? Ans. 15.

Ex. 6. There is a certain number, consisting of two places of figures, which, when divided by the sum of its digits, gives a quotient greater by 2, than the left hand digit; but if the digits be inverted, and then divided by a number greater by unity than the sum of the digits, the quotient will be greater by 2 than the preceding quotient. Required the number. Ans. 24.

Ex. 7. A gentleman has a garden in the form of an oblong, whose length is 20 yards and breadth 15. He wishes to have a walk made round it, whose area shall be $\frac{1}{4}$ of that of the whole garden. What must be the breadth of the walk? Ans. 3.44 Feet.

Ex. 8. A person bought a horse for a certain sum. He afterwards sold it for L. 144, and gained exactly as much per cent. as the horse cost him. Required the cost.

Ans. L. 80.

ON RATIOS AND PROPORTION.

95. By the term RATIO, is meant the relation which one quantity bears to another, with respect to magnitude. It is obvious, that this relation can exist only between quantities of the same kind, as of a line to a line, a surface to a surface, &c.

96. One quantity may be compared with another, either with a view of finding out their difference, or of finding how often the one contains the other. The relation which one quantity bears to another, with respect to their difference, is called their *Arithmetical* ratio. If the object of the comparison be to discover what multiple, part, or parts, the one is of the other, this relation is called their *Geometrical* ratio. In this latter sense, the term *ratio* is in the mean time to be used.

97. In order to discover the ratio subsisting between two quantities, the natural process is to divide the one by the other; in which case, the quotient will evidently express the fraction or multiple which the one is of the other. Thus, in comparing 2 with 6, we observe that the latter contains the former 3 times, $\frac{6}{2}$ being $=3$. Hence, then, 3 expresses the relative magnitude of 2 and 6, *meaning* that 6 is 3 times as great as 2; or, which amounts to the same thing, that 2 is $\frac{1}{3}$ part of 6. In the same manner, the fractions which express the ratio or relative magnitude of 4 and 6, 5 and 6, and 8 and 10, are $\frac{4}{6}, \frac{5}{6}$ and $\frac{8}{10} = \frac{2}{3}, \frac{5}{6}$ and $\frac{4}{5}$ respectively. Or, in general, the ratio of a to b is expressed by the fraction $\frac{a}{b}$. The ratio of one quantity to another, as of a to b, is expressed by placing two points between them; thus, $a:b$; where the former, from the circumstance of its standing first, is called the *antecedent* of the ratio, while the latter, from its following the other, is denominated the *consequent*.

98. If the terms of a ratio be multiplied or divided by the same quantity, the ratio is not altered. Thus, the ratio of a to b is the same with that of ma to mb, because $\frac{a}{b}=\frac{ma}{mb}$, (No. 42. Arith.) Hence, a ratio will be reduced to its lowest terms, by dividing both its terms by their greatest common measure.

99. That ratio is the greater, whose antecedent is the greater multiple part or parts of its consequent. Thus, the ratio of $3:4$ is greater than the ratio of $2:3$; because $\frac{3}{4}$ or $\frac{9}{12}$ is greater than $\frac{2}{3}$ or $\frac{8}{12}$.

100. A ratio is called a ratio of *less inequality*, of *equality*, or of *greater inequality*, according as the antecedent is *less* than, *equal to*, or *greater* than the consequent. Thus, $3:4$ is a ratio of less inequality; $4:4$ of equality; and $5:3$ of greater inequality.

101. A ratio of less inequality is increased, and of greater inequality diminished, by adding any quantity to both its terms. Thus, if to the terms of the ratio $3:5$, 2 be added, it becomes the ratio of $5:7$, which is a greater ratio than the former, because $\frac{5}{7}$ is greater than $\frac{3}{5}$. Or, to give a general demonstration, let $a:a+b$ represent any ratio of less inequality; let c be added to both terms, then the ratio becomes that of $a+c:a+b+c$. Now, the ratio of $a:a+b$, is otherwise expressed by $\frac{a}{a+b}$; and that of $a+c:a+b+c$ by $\frac{a+c}{a+b+c}$. Let now the two fractions $\frac{a}{a+b}$ and $\frac{a+c}{a+b+c}$ be reduced to a common denominator, and they become $\frac{a^2+ab+ac}{a^2+2ab+b^2+ac+bc}$ and $\frac{a^2+ab+ac+bc}{a^2+2ab+b^2+ac+bc}$. Now, since the numerator of the latter fraction is greater than that of the

former, it follows, that $\frac{a+c}{a+b+c}$ is greater than $\frac{a}{a+b}$, or, that $a+c:a+b+c$ is a greater ratio than that of $a:a+b$, and hence that the ratio of $a:a+b$ has been increased by adding the same quantity to both its terms. In the same manner, it might be shown, that any ratio of greater inequality, as that of $a+b:a$, is diminished by adding the same quantity c to both its terms; that a ratio of less inequality is *diminished*, while a ratio of greater inequality is *increased*, by *subtracting* the same quantity from both its terms.

102. Ratios are compounded together, by multiplying their antecedents together for a new antecedent, and their consequents together for a new consequent. Thus, if the ratios of $a:b$ and $c:d$ be compounded together, the resulting ratio is that of $ac:bd$. If the ratios of $3:4$, $5:6$, and $7:10$, be compounded together, the resulting ratio is that of $3\times5\times7:4\times6\times10=105:240=$ (dividing both by 15) $7:16$, (No. 98.)

103. If any ratio $a:b$ be compounded with itself *once*, *twice*, *thrice*, &c., the resulting ratios are those of $a^2:b^2$, $a^3:b^3$, $a^4:b^4$, &c. The ratio of $a^2:b^2$, is called the *duplicate* of $a:b$; $a^3:b^3$, the *triplicate*; $a^4:b^4$, the *quadruplicate*, &c. Hence the duplicate ratio of two quantities is the same as that of their *squares*; the triplicate, as that of their *cubes*, &c. &c. These ratios evidently receive their names from the indices of a and b; in this way, the ratio of $\sqrt{a}:\sqrt{b}$ is called the *sub-duplicate* of $a:b$; $\sqrt[3]{a}:\sqrt[3]{b}$, the *sub-triplicate*, &c. &c.

104. If any number of ratios, of which the consequent of the preceding ratio is the same with the antecedent of the succeeding one, be compounded together, the resulting ratio is that of the first antecedent to the last consequent. Thus, if the ratios $a:b$, $b:c$, $c:d$, &c. be compounded together, the resulting ratio is that of abc &c. : bcd &c. or (dividing both by bc) that of $a:d$,

105. A ratio of less inequality, compounded with another, diminishes it, while a ratio of greater inequality, compounded with another, increases it. Thus, let $x:y$ be com-

pounded with the ratio of $a : b$, the resulting ratio is that of $ax : by$, which is less or greater than $a : b$, according as $\frac{ax}{by}$ is less or greater than $\frac{a}{b}$, that is, according as x is less or greater than y.

106. If the difference between the antecedent and consequent of a ratio be small, when compared with either of them, the ratio of their squares will be nearly obtained by doubling this difference.

Let $a+x : a$ be a ratio where x is small when compared with a; then $a^2+2ax+x^2 : a^2$ is the *true* ratio of their squares; but as x, by supposition, is small in comparison with a, x^2 will be much smaller when compared with a^2+2ax, it may therefore be left out without much error; and hence $a^2+2ax : a^2$, or (dividing by a) $a+2x : a$, will very nearly express the ratio of the squares of $a+x$ and a.

Thus, an approximation to the duplicate ratio of $1001 : 1000$, is that of $1002 : 1000$.

In the same manner, it might be shown that the ratios of $a+3x : a$, $a+4x : a$, or, in general, of $a+mx : a$, are nearly equal to the ratios of $(a+x)^3 : a^3$, $(a+x)^4 : a^4$, and $(a+x)^m : a^m$, if x be small in comparison with a.

ON PROPORTION.

107. Proportion consists in the equality of ratios. If the ratio of $a : b$ be the same with that of $c : d$, or, which comes to the same thing, if $\frac{a}{b}=\frac{c}{d}$, then these four quantities are called *proportionals*, and are commonly written thus, $a : b = c : d$, or more generally thus, $a : b :: c : d$. This is expressed in language, by saying, that a is to b, as c to d. The first and last terms a and d are called the *extremes*, and the two middle terms, b and c, the *means*.

108. *If four quantities be proportionals, the product of the extremes is equal to the product of the means.*

For, if $a : b :: c : d$, then (No. 107.) $\frac{a}{b}=\frac{c}{d}$, and hence, by multiplying both sides of the equation by bd, we have $ad=bc$.

From this, it is manifest, that if any three terms of a proportion or analogy (as it is often called) be given, the fourth may be found. For, since $ad = bc$, $a = \frac{bc}{d}$, $d = \frac{bc}{a}$, $b = \frac{ad}{c}$, and $c = \frac{ad}{b}$.

109. *From the preceding theorem, it follows, that if three quantities be continually proportional, that is, if the first be to the second, as the second to the third, the product of the extremes is equal to the square of the mean.*

For, if $a : b :: b : c$, then (No. 108.) $ac = b^2$.

Hence, a mean proportional between two quantities is equal to the square root of their product. For, let x be a mean proportional between a and c, so that $a : x :: x : c$, then is $x^2 = ac$, and, therefore, $x = \sqrt{ac}$.

110. The converse of each of the preceding theorems is also true, that is, *If the product of two quantities be equal to the product of two others, or if the product of two be equal to the square of one, then the* FOUR, *or the* THREE *quantities will constitute an an analogy, provided the terms of the one product be made the extremes, and the terms of the other the means.*

For, if $ad = bc$, then, dividing both by bd, we shall have $\frac{a}{b} = \frac{c}{d}$, and hence, (No. 107.) $a : b :: c : d$.

In like manner, if $ac = b^2$, then, dividing both by bc, we have $\frac{a}{b} = \frac{b}{c}$, and hence, as before, $a : b :: b : c$.

111. *Ratios that are equal to the same ratio, are equal to one another*, that is, if $a : b :: c : d$, and $c : d :: e : f$, then shall $a : b :: e : f$.

For $\frac{a}{b} = \frac{c}{d}$, and $\frac{c}{d} = \frac{e}{f}$, hence, $\frac{a}{b} = \frac{e}{f}$, and, consequently, (No. 107.) $a : b :: e : f$.

112. *If four quantities be proportionals, they are also proportionals when taken* alternately, that is the first : third :: second : fourth.

For, if $a:b::c:d$, then (No. 108.) $ad=bc$, and dividing both by cd, we have, $\frac{a}{c}=\frac{b}{d}$, and, therefore, (No. 107.) $a:c::b:d$.

113. *If four quantities be proportionals, they are also proportionals when taken* inversely, *that is the* second : first : : fourth : third.

For, since $a:b::c:d$, $bc=ad$, (No. 108.) and dividing both by ac, we have $\frac{b}{a}=\frac{d}{c}$, and, therefore, (No. 107.) $b:a::d:c$.

114. *If four quantities be proportionals, they are also proportionals* componendo *or* dividendo, *that is, the sum or difference of the first and second, is to the second, as the sum or difference of the third and fourth to the fourth.*

For, since $a:b::c:d$, $\frac{a}{b}=\frac{c}{d}$, and, therefore, $\frac{a}{b}\pm 1=\frac{c}{d}\pm 1$, or, reducing to improper fractions, $\frac{a\pm b}{b}=\frac{c\pm d}{d}$, and hence, (No. 107.) $a\pm b:b::c\pm d:d$.

115. *If four quantities be proportionals, they are also proportionals* convertendo, *that is, the first is to its excess above the second, as the third to its excess above the fourth.*

For, since $a:b::c:d$, $a-b:b::c-d:d$, (No. 114.) and, alternately, (No. 112.) $a-b:c-d::b:d$; and also from alternation, $a:c::b:d$, and hence (No. 111.) $a:c::a-b:c-d$, and, therefore, by again alternating, $a:a-b::c:c-d$.

116. *When four quantities are proportionals, the sum of the first and second is to their difference, as the sum of the third and fourth to their difference.*

This readily appears from Nos. 114 and 115.

For, (No. 114.) $a+b:b::c+d:d$, therefore, (No. 112.) $a+b:c+d::b:d$. And, (No. 115.) $a:a-b::c:c-d$, therefore, (No. 112.) $a:c::a-b:c-d$. But, (No. 112.) $a:c::b:d$, hence, (No. 111.) $a+b:c+d::a-b:c-d$; and, therefore, alternately, $a+b:a-b::c+d:c-d$.

117. *When any number of quantities are proportionals,*

as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $a:b::c:d::e:f$, &c. Then shall $a:b::a+c+e+\&c.:b+d+f+\&c.$

For, since $a:b::c:d$, $ad=bc$, (No. 108.;) in like manner, because $a:b::e:f$, $af=be$, now, $ab=ba$; and, therefore, by adding equals to equals, we have $ab+ad+af+\&c.=ba+bc+be+\&c.$, or, resolving into factors, $a(b+d+f+\&c.)=b(a+c+e+\&c.)$ and hence, (No. 110.) $a:b::a+c+e+\&c.:b+d+f+\&c.$

118. *When four quantities are proportionals, if the first and second be multiplied or divided by any quantity, as also the third and fourth, the resulting quantities will be proportionals.*

For, if $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$, and $\frac{ma}{mb}=\frac{nc}{nd}$.

Hence, $ma:mb::nc:nd$, when m and n may be any quantities whatever, either integral or fractional.

From the preceding theorem, it follows, that if the first and third, as also the second and fourth, be multiplied or divided by any quantity, the resulting quantities will be proportionals.

119. *In two ranks of proportionals, if the corresponding terms be multiplied together, the resulting products will be proportionals.*

$$\begin{array}{l} \text{If } a:b::c:d \\ \text{And } e:f::g:h \\ \hline \text{Then } ae:bf::cg:dh. \end{array}$$

For, since $a:b::c:d$, $\frac{a}{b}=\frac{c}{d}$, and, in like manner, $\frac{e}{f}=\frac{g}{h}$; hence, $\frac{a}{b}\times\frac{e}{f}=\frac{c}{d}\times\frac{g}{h}$, that is, $\frac{ae}{bf}=\frac{cg}{dh}$, and, therefore, (No. 107.) $ae:bf::cg:dh$.

The same demonstration may be applied to any number of ranks of proportional quantities.

120. *If four quantities be proportionals, any like powers or roots of them will also be proportionals.*

For, since $a:b::c:d$, $\frac{a}{b}=\frac{c}{d}$, and, therefore, $\frac{a^n}{b^n}=\frac{c^n}{d^n}$, hence, (No. 107.) $a^n:b^n::c^n:d^n$, where n may be either integral or fractional.

121. *If there be any number of quantities*, a, b, c, d, *&c. in continued proportion, the first is to the third in the duplicate ratio of the first to the second; or the first to the fourth in the triplicate ratio of the first to the second, &c.*

For, since $a:b::b:c$, $a^2:b^2::b^2:c^2$, (No. 120.) but $ac=b^2$, (No. 108.) therefore, $a^2:b^2::ac:c^2$, and hence, dividing the two last terms by c, we have $a:c::a^2:b^2$, (No. 118.)

Again, because $a:c::a^2:b^2$.
And, $c:d::a:b$.

We have, (No. 119.) $ac:cd::a^3:b^3$, or, dividing the two first terms by c, $a:d::a^3:b^3$.

In like manner, it may be shown, that $a:e::a^4:b^4$, &c. &c.

122. In the preceding theorems, it is supposed that one quantity is some determinate multiple, part, or parts of another, or that the fraction arising from the division of the one by the other, is some determinate fraction. This will always be the case when the two quantities have any common measure whatever.

Let x be a common measure of a and b, and let $a=mx$, $b=nx$; then $\frac{a}{b}=\frac{mx}{nx}=\frac{m}{n}$, when m and n are whole numbers.

123. It often happens, however, that the quantities are *incommensurable*, or have no common measure whatever, as when one represents the side of a square, and the other its diagonal. In such cases, the value of $\frac{a}{b}$ cannot be expressed exactly by any fraction $\frac{m}{n}$, whose numerator and denominator are whole numbers; a fraction, however, of this kind, may be found, which will express its value to any degree of accuracy less than perfect.

Suppose x to be a measure of b, and let $b=nx$; also, let a be greater than mx, but less than $(m+1)x$; then $\frac{a}{b}$ is greater than

$\frac{m}{n}$ but less than $\frac{m+1}{n}$, or the difference between $\frac{m}{n}$ and $\frac{a}{b}$, is less than $\frac{1}{n}$; and as x is diminished, n will be increased, since $nx = b$, and hence $\frac{1}{n}$ diminished. By diminishing x, therefore, the difference of $\frac{a}{b}$ and $\frac{m}{n}$ may be rendered less than any that can be assigned.

124. If a and b, as well as c and d, be incommensurable, and if, when $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, $\frac{c}{d}$ lie also between $\frac{m}{n}$ and $\frac{m+1}{n}$, however the magnitudes m and n are increased, $\frac{a}{b} = \frac{c}{d}$. If they are not equal, they must have some assignable difference; and because each of them lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, this difference is less than $\frac{1}{n}$, but since, by supposition, n may be increased without limit, $\frac{1}{n}$ may be diminished without limit, that is, it may become less than any assignable quantity, therefore, $\frac{a}{b}$ and $\frac{c}{d}$ have no assignable difference, that is, $\frac{a}{b} = \frac{c}{d}$; and all the preceding propositions, respecting proportionals, are true of the four magnitudes a, b, c, and d.*

ON ARITHMETICAL PROGRESSION.

125. Quantities are said to be in *Arithmetical Progression*, when they increase or decrease by a common difference.

Thus, 1, 3, 5, 7, 9, 11, &c.; 5, 3, 1, — 1, — 3, — 5, &c. &c., are quantities in arithmetical progression.

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* Wood's Algebra, Art. 191.

126. If a be the first term, and b the common
of a series of terms in arithmetical progression, tl
cond term will be $a+b$, the *third* $a+2b$, the *fou*
&c.; and since the coefficient of b in any term
unity than the number of terms, counting from t
ning, the *last* term will therefore be $a+(n-1)l$
denotes the number of terms.

127. *The sum of a series of quantities, in Ari*
Progression, is equal to the sum of the first and l
or of any two terms equally distant from them, r
by half the number of terms.

Let $s=$ sum of the series a, $a+b$, $a+2b$, $a+3b$, &c.
Then, $s=a$ - - - - - $+a+b$ - - - $+a+2b$, &c. - - $a+$
And $s=a+(n-1)b+a+(n-2)b+a+(n-3)b$, &c.
Or, $2s=2a+(n-1)b+2a+(n-1)b+2a+(n-1)b+$
terms, by adding the two series together.
Hence. $2s=(2a+(n-1)b)\times n$, and, therefore, s

$$(n-1)b)\frac{n}{2}.$$

From the equation $s=(2a+(n-1)b)\frac{n}{2}$, any three of
quantities s, a, b and n, being given, the fourth may b
It will be fully as well, however, to substitute the giv
bers in the general *formula*, $s=(2a+(n-1)b)\frac{n}{2}$, and t
equation determine the value of the unknown quantity.

128. *If three quantities be in arithmetical progres*
sum of the extremes is double of the mean.

For, if a, $a+b$, $a+2b$, be the three quantities, then i
vious, that $a+(a+2b)=(a+b)2$.
Hence, an arithmetical mean between two quantities i
to half their sum.

Ex. 1. Required the sum of 20 terms of the series 1, 3
9, &c.

Here, $a=1$, $b=2$, and $n=20$; hence, $s=(2a+(n-$

$$=(2+(20-1)2)\frac{20}{2}=(2+38)10=40\times10=400.$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] travels [illegible] his journey [illegible] at the rate of [illegible] and so on. [illegible]

Ans. [illegible]

GEOMETRICAL PROGRESSION

[illegible]en a series of [illegible] is formed from [illegible] [illegible]plication of the [illegible] quantity, by the [illegible] [illegible] quantities are [illegible] to be in [illegible]

Thus, 1, 2, 4, 8, &c.; 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., are quantities in geometrical progression.

The constant multiplier is called the *common ratio*, and is manifestly equal to the quotient arising from the division of any term, by that which immediately precedes it.

130. If a be the first term, and r the common ratio of a series of terms in geometrical progression, then the second term will be ar, the third ar^2, the fourth ar^3, &c., or, since the index of r in any term is less by unity than the number of terms, reckoning from the beginning, the last term will be ar^{n-1}, when n denotes the number of terms.

Hence, if l denote the last term, we shall have $l = ar^{n-1}$.

131. *The sum, of a series of quantities in geometrical progression, may be found, by subtracting the first term from the product of the last term and common ratio, and dividing the remainder by the difference between the common ratio and unity.*

Let s = sum of the series.

Then, $s = a+ar+ar^2 \cdots +ar^{n-2}+ar^{n-1}$, and multiplying by r, $rs = ar+ar^2+ar^3 \cdots +ar^{n-1}+ar^n$, subtract the former, $s = a+ar+ar^2+ar^3 \cdot \quad ar^{n-1}$.

Rem. $rs - s = -a \cdots\cdots\cdots +ar^n$.

Hence, $(r-1)s = ar^n - a$, and, therefore, $s = \frac{ar^n - a}{r-1}$. But, $l = ar^{n-1}$, therefore, $rl = ar^n$, and hence, by substitution, $s = \frac{rl - a}{r-1}$.

If r is a proper fraction, then r and its powers are less than unity, and, consequently, both numerator and denominator of the fraction $\frac{rl-a}{r-1}$, would become negative. The division of the one by the other would no doubt give a positive result, but it will be rather more convenient to transform the equation $s = \frac{rl-a}{r-1}$, into the form $s = \frac{a-rl}{1-r}$, by multiplying both terms by -1.

132. When r is a proper fraction, and n indefinitely great, ar^n becomes indefinitely small; and, therefore, $s = \frac{a}{1-r}$, the *limit* to which the sum of the series approaches.

133. From the equation $s = \frac{rl-a}{r-1}$, any three of the four quantities s, r, l and a being given, the fourth may be found.

EXAMPLES.

Ex. 1. Required the sum of 8 terms of the series 1, 2, 4, 8, &c.

Here, $a = 1$, $r = 2$, and $n = 8$.

Now, $s = \frac{ar^n - a}{r-1} = \frac{1 \times 2^8 - 1}{2-1} = \frac{256-1}{1} = 255.$

Ex. 2. Required the sum of 6 terms of the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c.

$$s = \frac{a - ar^n}{1-r} = \frac{\frac{1}{2} - \frac{1}{2} \times \frac{1}{2})^6}{1-\frac{1}{2}} = \frac{\frac{1}{2} - \frac{1}{128}}{\frac{1}{2}} = \frac{\frac{63}{128}}{\frac{1}{2}} = \frac{126}{128} = \frac{63}{64}.$$

When n is a large number, the solution of the fundamental equation is most conveniently effected by *Logarithms*, as will afterwards appear.

Ex. 3. Required the value of the repeating decimal .3333.

This decimal is represented by the geometric series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} +$ &c., whose first term is $\frac{3}{10}$, and common ratio $\frac{1}{10}$, and in this case n is indefinitely great.

Hence, $s = \frac{a}{1-r} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{3}{10} \times \frac{10}{9} = \frac{30}{90} = \frac{3}{9} = \frac{1}{3}.$

EXERCISES.

Ex. 1. Required the sum of the series 1, 2, 4, 8, 16, &c. to 14 terms. Ans. 16383.

Ex. 2. Find the value of the series, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. continued indefinitely. Ans. $\frac{3}{2}$.

Ex. 3. Required the value of the circulating decimal $0.\dot{4}\dot{5}$.

Ans. $\frac{5}{11}$.

Ex. 4. A person sold a horse, on condition of his receiving a farthing for the first nail in his shoes, a halfpenny for the second, a penny for the third, and so on, doubling the price of every nail, to 32, the number of nails in the four shoes. Required the price of the horse. Ans. L. 4,473,924, 5s. 3¾d.

ELEMENTS
OF
GEOMETRY.

BOOK I.

DEFINITIONS.

I.—A POINT is that which has position but not magnitude.

II.—A line is length without breadth.

Corollary. The extremities of a line are points; and the intersections of one line with another are also points.

III.—If two lines are such, that they cannot coincide in any two points, without coinciding altogether, each of them is called a straight line.

Cor.—Hence, two straight lines cannot inclose a space. Neither can two straight lines have a common segment; for they cannot coincide in part, without coinciding altogether.

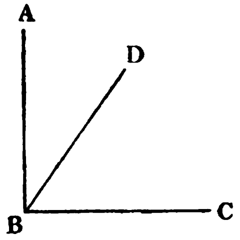
IV.—A superficies is that which has only length and breadth.

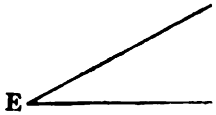
Cor.—The extremities of a superficies are lines, and the intersections of one superficies with another, are also lines.

V.—A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

Book I.

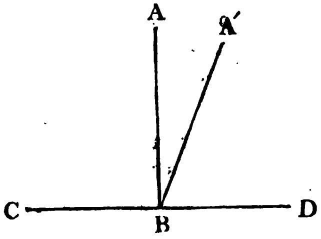
VI.—A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.





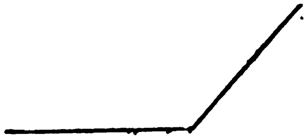
N. B.—When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line. Thus, the angle which is contained by the straight lines AB, CB, is named the angle ABC, or CBA; that which is contained by AB, DB, is named the angle ABD, or DBA; and that which is contained by BD, CB, is called the angle DBC, or CBD; but, if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at E.

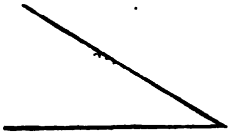
VII.—When a straight line AB, standing on another straight line CD, makes the adjacent angles ABC and ABD equal to one another, each of the angles is called a right angle, and the straight line, which stands on the other, is called a perpendicular to it.



Cor. The angles which one straight line AB makes with another CD, upon one side of it, are either two right angles, or are, together, equal to two right angles. For, if AB be perpendicular to CD, then each of the angles ABC, ABD, is a right angle; and, therefore, are, together, equal to two right angles. But, if one of them, A′BC, exceed a right angle, the other, A′BD, is manifestly as much less than a right angle, and, consequently, the two are together equal to two right angles.

VIII.—An obtuse angle is that which is greater than a right angle.





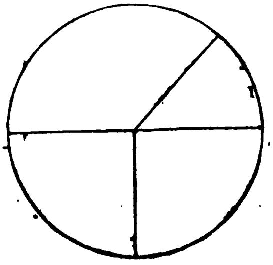
IX.—An acute angle is that which is less than a right angle.

X.—A figure is that which is inclosed by one or more boundaries.

XI.—Figures are said to be equivalent, when, without coinciding, they contain the same space.

XII.—A circle is a plane figure, described by the revolution of a straight line, about one of its extremities, which remains fixed.

XIII.—The fixed point is called the *centre* of the circle, the describing line its *radius*, and the boundary traced by the remote end of that line its *circumference*.



Book I.

XIV.—The diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

Cor. All radii of the same circle are equal to each other, and to a semi-diameter.

XV.—A semicircle is the figure contained by a diameter, and the part of the circumference cut off by the diameter.

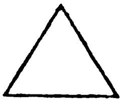
XVI.—Rectilineal figures are those which are contained by straight lines.

XVII.—Trilateral figures, or triangles, by three straight lines.

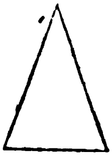
XVIII.—Quadrilateral, by four straight lines.

XIX.—Multilateral figures, or polygons, by more than four straight lines.

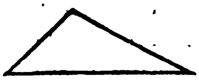
XX.—Of three sided figures, an equilateral triangle is that which has three equal sides.



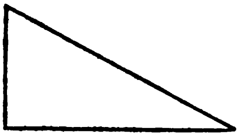
XXI.—An isosceles triangle, is that which has two sides equal.



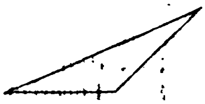
XXII.—A scalene triangle, is that which has three unequal sides.



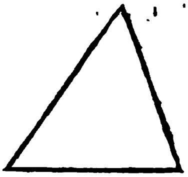
XXIII.—A right angled triangle, is that which has a right angle.



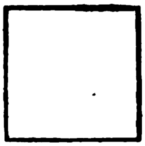
XXIV.—An obtuse angled triangle, is that which has an obtuse angle.



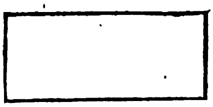
XXV.—An acute angled triangle, is that which has three acute angles.



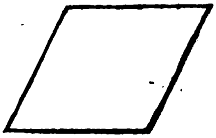
XXVI.—Of four sided figures, a square is that which has all its sides equal, and all its angles right angles.



XXVII.—An oblong, is that which has all its angles right angles, but has not all its sides equal.



XXVIII—A rhombus, is that which has all its sides equal, but its angles are not right angles.



XXIX.—A rhomboid, is that which has its opposite sides equal to one another.

XXX.—All other four sided figures besides these, are called Trapegiums.

XXXI.—Straight lines, which are in the same plane, and being produced even so far both ways, do not meet, are called *Parallel* lines.

A Proposition, is the enunciation of something for construction, or demonstraction, and is either a Problem or Theorem.

In a Problem, something is required to be performed.

In a Theorem, some truth is to be demonstrated.

A *Lemma* is a subsidiary truth, employed in the demonstration of a theorem, or the solution of a Problem.

A *Corollary* is an obvious consequence that arises from a proposition.

A *Scholium* is a remark on the nature and application of a proposition.

An *Axiom* is a self-evident truth.

AXIOMS.

I.—Things which are equal to the same thing, are equal to one another.

II.—If equals be added to equals, the wholes are equal.

III.—If equals be taken from equals, the remainders are equal.

IV.—If equals be added to unequals, the wholes are unequal.

V.—If equals be taken from unequals, the remainders are unequal.

VI.—Things which are doubles of the same thing, are equal to one another.

VII.—Things which are halves of the same thing, are equal to one another.

VIII.—Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.—The whole is greater than its part, and equal to all its parts taken together.

X.—All right angles are equal to one another.

XI.—Two straight lines, which intersect one another, cannot be both parallel to the same straight line.

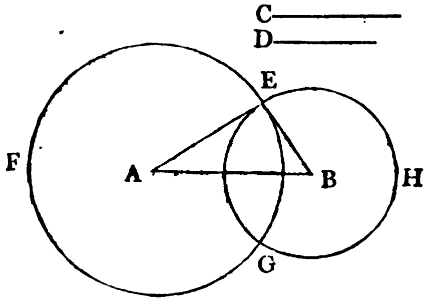
XII.—The shortest distance between two points is a straight line.

Cor. Any two sides of a triangle are together greater than the third side.

PROPOSITION I. PROBLEM.

To construct a triangle, of which the sides shall be equal to three given straight lines; any two of which are greater than the third.[a] a Cor. Ax. 12.

Let AB represent the base, and C, D, the two sides of the triangle which it is required to construct. From the centre A, with the distance C, describe the circle EFG, and from the centre B, with the distance D, describe the circle EGH, meeting the former in E; join AE and EB; AEB is the triangle required.



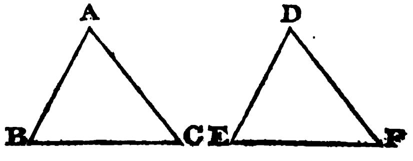
Because the circle EFG is described with a radius equal to C, therefore, AE[b] is équal to C; and, b Cor. 14. for the same reason, BE is equal to D. Consequently, the Def. triangle AEB answers the conditions of the problem.

Cor. If the radii C and D be equal to each other, the triangle will evidently be isosceles; and if the three lines are equal to one another, the triangle will be equilateral.

PROP. II. THEOREM.

If two triangles, ABC, and DEF, have the two sides BA, AC of the one, equal to the two sides ED, DF of the other, each to each, and have likewise the angles BAC and EDF, contained by those sides, equal, the triangles are equal in every respect, and have those angles equal, which are opposite to the equal sides.

For, if the triangle ABC be applied to the triangle DEF, so that the point A may be on D, and the side AB lie along

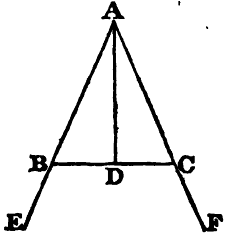


the side DE; then the point B shall coincide with E, because AB is equal to DE[s]; now, AB coinciding with DE, AC shall fall along DF, because the angle BAC is equal to the angle EDF[s]; wherefore, also the point C shall coincide with the point F, because the side AC is equal to the side DF[s]; and because B coincides with E, and C with F, the side BC coincides with the side EF, and is therefore equal to it. Hence, also, the whole triangle ABC coincides with the whole triangle DEF, so that the spaces which they contain, or their areas, are equal. Now, since while BA coincides with ED, BC also coincides with EF, it follows that the angle ABC is equal to the angle DEF, and these are opposite to the equal sides AC and DF; in the same manner, it appears that the angle ACB is equal to the angle DFE, and they are opposite to the equal sides AB and DE. Therefore, if two triangles, &c. Q. E. D.

s Sup.

PROP. III. THEOREM.

In an Isoceles triangle ABC, the angles ABC and ACB opposite to the equal sides AC and AB are equal.



For, let the straight line AD bisect or divide the vertical angle BAC into two equal angles, BAD and DAC, and meet BC in the point D; then, because AB[s] is equal to AC[s] and AD common to the two triangles ABD and ACD, and likewise the included angle BAD equal to the included angle CAD[a], these triangles are equal in every respect[b]; and, therefore, the angle ABD is equal to the angle ACD.

a Cons.
b 2. 1.

Cor. 1. If two angles of a triangle be equal, the sides which subtend, or are opposite to them, are also equal.

Cor. 2. Every equilateral triangle is also equiangular.

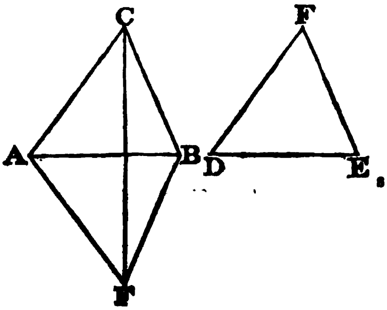
Cor. 3. The straight line which bisects the vertical angle of an isosceles triangle, also bisects the base at right angles.

Cor. 4. If the equal sides AB and AC be produced to E and F, the angles EBC and BCF on the other side of the base are likewise equal.

Book I.

PROP. IV. THEOREM.

If two triangles, ABC and DEF, have the three sides of the one equal to the three sides of the other, each to each, viz. AB to DE, AC to DF, and BC to EF, the triangles are equal in every respect.



Let AB be that side which is not less than any other side of the triangle. Then, if the base DE be applied to the base AB, they will coincide, because AB is equal to DE[s]. Let s Sup. the two sides DF, FE fall upon the other side of the base AB from that on which the triangle ABC is; join CF. Since AC is equal to AF[s], the angle ACF is equal to the angle AFC[a]; and, in the a 3. 1. same manner, because BC is equal to BF[s], the angle BCF is equal to the angle BFC[a]; therefore, the whole angle ACB is equal to the whole angle AFB or DFE. Now, the two sides AC, CB, which contain the angle ACB, are equal to the two sides DF, FE[s], which contain DFE, therefore the triangle ACB is equal in every respect to the triangle DFE[b]; consequently, b 2. 1. the remaining angles in the one are equal to the remaining angles in the other, each to each, to which the equal sides are opposite, viz. the angle CAB equal to FDE, and CBA to FED. Therefore, if two triangles, &c. Q. E. D.

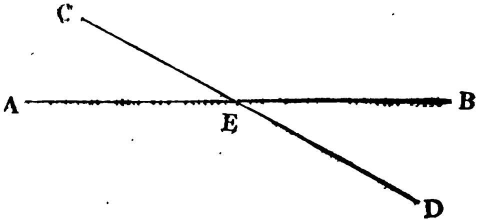
Cor. 1. The straight line drawn from the vertex of an isosceles triangle, so as to bisect the base, is perpendicular to it, and likewise bisects the vertical angle. Hence a convenient method of drawing a perpendicular to a given straight line from a given point, either in or without the given line.

Cor. 2. If AC and CB were equal, as also AF and FB, then it would follow, that the angles ACF and BCF were also equal; or that the angle ACB would be bisected, as also the straight line AB. Hence an obvious method of bisecting a given angle or a given straight line.

Scholium. From what is demonstrated in the proposition, we can, by the help of Prop. I., make an angle at any given point in a straight line, equal to any given angle. Let ACB be any given angle, and let it be required at the point F in the straight line FD, to make an angle equal to ACB. Take any two points A, B, in the lines which contain the angle ACB, and join AB. Make the triangle FDE, having its three sides respectively equal to those of ACB[c], viz. FD to CA, FE to CB, and DE to AB. Then is the angle DFE equal to ACB. c 1. 1.

PROP. V. THEOREM.

If two straight lines AB and CD cut one another in E, the vertical or opposite angles AEC and BED are equal.



For the angles CEA and CEB, which CE makes with AB, are together equal to two right angles[a]; and the angles CEB and BED, which BE makes with CD, are also together equal to two right angles[a]; therefore, the two angles CEA and CEB are equal to the two CEB and BED[b]; take away the common angle CEB, and the remaining angle CEA is equal to the remaining angle BED[c]. In the same manner it may be demonstrated that the angles CEB and AED are equal. Therefore, if two straight lines, &c. Q. E. D.

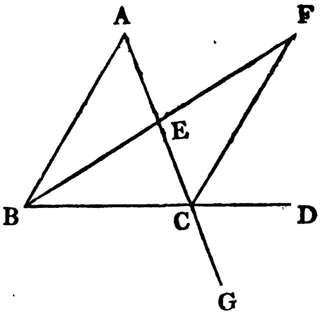
a Cor. Def. 7.
b Ax. 1.
c Ax. 3.

Cor. 1. From this it is manifest, that if two straight lines cut one another, the angles which they make at the point of the intersection are together equal to four right angles.

Cor. 2. And hence, all the angles made by any number of straight lines, meeting in one point, are together equal to four right angles.

PROP. VI. THEOREM.

If one side BC of any triangle, ABC, be produced to D, the exterior angle ACD is greater than either of the interior and opposite angles CBA, CAB.



Bisect AC[a] in E, join BE, and produce it to F, making EF equal to EB; join also FC, and produce AC to G. a Cor.2.4.1.

Because AE is equal to EC[g], and BE to EF[g], the two sides AE, EB are equal to the two CE, EF, each to each; and the included angle AEB is equal to the included angle, CEF being vertical angles[b]; therefore, the two triangles AEB and CEF are equal in every respect[c]; consequently, the angle BAE is equal to the angle ECF; but the angle ECD is greater than the angle ECF[d]; therefore, the angle ECD, that is, ACD is also greater than the angle BAE, or BAC. In the same manner, if the side BC be bisected, it may be demonstrated that the angle BCG, that is, ACD[e], is greater than ABC. Therefore, if one side, &c. Q. E. D.

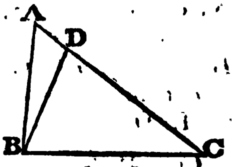
g Cons.
b 5. 1.
c 2. 1.
d Ax. 9.
e 5. 1.

Cor. Any two angles of a triangle, are together less than two right angles. For, since by the proposition, the exterior angle ACD is greater than the interior and opposite angle ABC, if ACB be added to each, then ACD and ACB will be greater than ABC and ACB, but ACD and ACB are together equal to two right angles[f]; wherefore, ABC and ACB are together less than two right angles. In like manner, it may be demonstrated that BAC, ACB, as also CAB, ABC, are together less than two right angles. f Cor. Def.7.

PROP. VII. THEOREM.

The greater side of every triangle has the greater angle opposite to it.

If the side CA be greater than CB, the angle CBA is



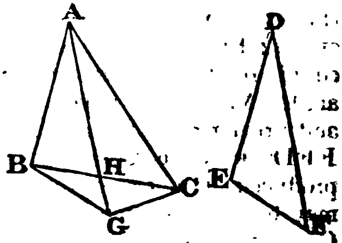
also greater than the angle CAB. From CA, which is greater than CB, cut off CD, equal to CB, and join BD. Then, because c Cons. CD is equal to CB[c], the angle CDB is a 3. 1. equal to the angle CBD[a]; but the angle CDB being the exterior angle of the triangle ABD, is greater than the b 6. 1. interior and opposite angle BAD[b]; consequently, the angle CBD is also greater than the angle BAD, much more, then, is the angle CBA greater than the angle BAD or CAB. Therefore, the greater side, &c. Q. E. D.

Cor. The greater angle of every triangle has the greater side opposite to it.

PROP. VIII. THEOREM.

If two triangles, ABC, and DEF, have two sides BA, AC of the one, equal to two sides ED, DF of the other, each to each, but the included angle, BAC of the one, greater than the included angle EDF of the other, the base BC of that which has the greater angle, shall be greater than the base EF of the other.

Of the two sides AB, AC, let AB be that which is not greater than the other, and, at the point A in the straight line AB, make the angle BAH equal to the angle a Scholium, 4. 1. EDF[a]. Then, since AB is not greater than AC, the angle ACB is not greater than ABC, but the exterior angle AHC is b 6. 1. greater than ABC[b], and, therefore, greater also than ACH; consequently, the side AC is c Cor. 6. 1. greater than AH[c]. Let AH be produced till AG is equal AC, or DF, and join BG, GC. The triangles BAG, EDF, d 2. 1. are equal[d], since the sides AB, AG are equal to the sides DE, DF, each to each, and the angle BAG equal to the angle EDF, therefore, the base BG is equal to the base EF. Now, since AG and AC are equal, the angles AGC, ACG e 3. 1. are likewise equal[e]; but the angle BGC is greater than the



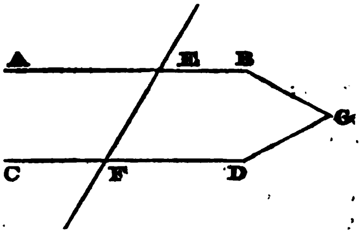
angle AGC[f], therefore, the angle BGC is greater also than ACG; much more, then, is the angle BGC greater than the angle BCG; therefore, the side BC is greater than BG[c]; and, as BG is equal to EF, the base BC is greater also than the base EF. Wherefore, if two triangles, &c. Q. E. D.

f Ax. 9.

Cor. If two triangles have two sides of the one respectively, equal to two sides of the other, but the base of the one greater than the base of the other, the angle contained by the two sides of that which has the greater base, shall be greater than the angle contained by the two sides of the other.

PROP. IX. THEOREM.

If a straight line EF, falling upon two other straight lines AB, CD, makes the alternate angles AEF, EFD equal to one another, these two straight lines are parallel.



For, if AB and CD are not parallel, they will meet, if produced, either towards A and C, or towards B and D; let them be produced, and meet towards B and D in the point G; then GEF is a triangle, and its exterior angle AEF is greater than the interior and opposite angle EFG[a]; but it is also equal to it[s], *that is*, the angle AEF is both equal and greater than the angle EFD, which is impossible. Therefore, AB and CD being produced, do not meet towards B and D. In like manner, it may be demonstrated, that they do not meet towards A and C; but those straight lines which meet neither way, though produced ever so far, are parallel to one another[b]. AB, therefore, is parallel to CD. Wherefore, if a straight line, &c. Q. E. D.

a 6. 1.
s Sup.
b Def. 31.

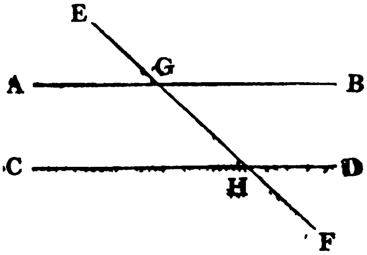
PROP. X. THEOREM.

If a straight line, EF, falling upon two other straight lines AB, CD, makes the exterior angle EGB, equal to the interior and opposite angle GHD upon the same side, or makes the two interior angles BGH, GHD, on the same side, together equal to two right angles, AB is parallel to CD.

a Sup.
c 5. 1.

Because the angle EGB is equal to the angle GHD[a], and also to the angle AGH[c], the angle AGH is equal to the angle GHD, and they are the alternate angles; therefore, AB is parallel to CD[a]. Again, because the angles BGH and GHD are equal to two right angles[s], and AGH, and BGH, are also equal to two right angles[c], the angles AGH, BGH, are equal to the angles BGH, GHD[b]; take away the common angle BGH; therefore, the remaining angle AGH, is equal to the remaining angle GHD[c], and they are alternate angles; therefore, AB is parallel to CD[d]. Wherefore, if a straight line, &c. Q. E. D.

a 9. 1.
s Sup.
c Cor. Def 7.
b Ax. 1.
c Ax. 3.
d 9. 1.

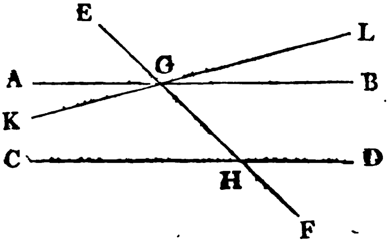


Scholium. This, and the preceding proposition, furnish the three *criteria* whereby to judge of the parallelism of straight lines.

PROP. XI. THEOREM.

If a straight line, EF, fall upon two parallel straight lines, AB, CD, it makes the alternate angles AGH, GHD, equal to one another; and the exterior angle EGB equal to the interior and opposite angle GHD upon the same side, and the two interior angles BGH and GHD upon the same side, together equal to two right angles.

For, if AGH be not equal to GHD, let HG be drawn, making the angle KGH equal to GHD, and produce KG to



L, then KL will be parallel to CD[a]; but AB is also parallel to CD[s]; therefore, two straight lines are drawn through the same point G, parallel to CD, and yet not coinciding with one another, which is impossible[b]. The angles AGH and GHD, are therefore not unequal, that is, they are equal to one another. Now, the angle EGB is equal to the angle to AGH[c] and AGH was proved to be equal to GHD, therefore, EGB is likewise equal to GHD; add to each of these the angle BGH, therefore, the angles EGB, BGH are equal to the angles BGH, GHD, but EGB, BGH, are together equal to two right angles[d]; therefore, also BGH, GHD are equal to two right angles. Wherefore, if a straight line, &c. Q. E. D.

Cor. If two lines, KL and CD, make with EF, the two angles KGH, GHC, together less than two right angles, KG and CH will meet on the side of EF, on which the two angles are, that are less than two right angles. For, if not, KL and CD are either parallel, or they meet in the other side of EF; but they are not parallel, for the angles KGH, GHC, would then be equal to two right angles[e]. Neither do they meet on the other side of EF, for the angles LGH, GHD, would then be two angles of a triangle, and less than two right angles[f]; but this is impossible, for the four angles KGH, HGL, CHG, GHD, are together equal to four right angles[g], of which the two, KGH, CHG are, by supposition, less than two right angles; therefore, the other two, HGL, GHD, are greater than two right angles. Therefore, since KL and CD are not parallel, and since they do not meet towards L and D, they must meet if produced towards K and C.

Cor. 2. Straight lines which are parallel to the same straight line, are parallel to one another.

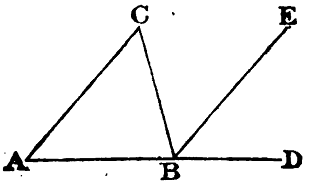
Scholium. This proposition suggests an easy method of drawing a parallel to a given straight line through a given point. Let G be the point, and CD the straight line. Take H any point in CD, join GH, make the angle BGH equal to GHD[h], and produce BG to A; then is AB parallel to CD.

[a] 9. 1.
[s] Sup.
[b] Ax. 11.
[c] 5. 1.
[d] Cor. Def. 7.
[e] 11. 1.
[f] Cor. 6. 1.
[g] Cor. Def. 7.
[h] Sch. 4. 1.

PROP. XII. THEOREM.

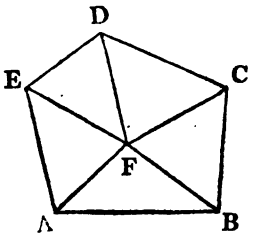
If a side AB of any triangle ABC be produced to D, the exterior angle CBD is equal to the two interior and opposite angles BAC, ACB, and the three interior angles of any triangle ABC, viz. ABC, BCA, CAB, are together equal to two right angles.

Through the point B draw BE parallel to the straight c Sch. 11. line AC[c]. Then, because AC is parallel to BE and CB meets them, the alternate angles a 11. 1. ACB, CBE are equal[a]. Again, because AC is parallel to BE, and AD falls upon them, the exterior angle EBD is equal to the interior and opposite angle CAB[a]; but the angle CBE was shown to be equal to the angle ACB; therefore, the whole exterior angle CBD is equal to the two interior and opposite angles CAB, ACB; to these equals, add the angle ABC, then the angles CBD, CBA will be equal to the three angles CBA, BAC, ACB; but the b Cor.7.Def. angles CBD, CBA are together equal to two right angles[b]; therefore, the angles CBA, BAC, ACB are also together equal to two right angles.



Cor. All the interior angles of any rectilineal figure are equal to twice as many right angles as the figure has sides, wanting four right angles.

For any rectilineal figure, ABCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles. And, by the preceding theorem, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure; and the same angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the tri- d Cor. 2. 5. angles, that is, together with four right angles[d]. Therefore,

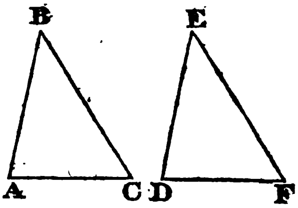


twice as many right angles as the figure has sides, are equal to all the angles of the figure, together with four right angles; that is, the angles of the figure are equal to twice as many right angles, as the figure has sides wanting four.

PROP. XIII. THEOREM.

If two triangles, BAC, EDF, have two angles, BAC, ABC, and a side AC in the one, equal to two angles EDF, DEF, and a corresponding side DF in the other, each to each, the triangles shall be equal in every respect.

For, since two angles in the one triangle are respectively equal to two angles in the other, the third angle ACB in the one is equal to the third angle DFE in the other[a]. Now, if the triangle ABC be applied to the triangle EDF, so that the point A may be on D, and AC lie along DF, the point C will also coincide with F, since AC is equal to DF[s]; but AC coinciding with DF, AB will fall along DE, because the angle CAB is equal to the angle FDE[s]; therefore, the point B will be found somewhere in the line DE. In the same manner, CB will fall along FE, because the angle ACB is equal to DFE, and the point B will therefore also be found somewhere in the line FE; hence the point B is situated at the intersection of DE and FE, and will therefore coincide with E. Thus, it appears, that the two triangles exactly coincide, and they are therefore equal in every respect[b]. Wherefore if two triangles, &c. Q. E. D.



a 12. 1.

s Sup.

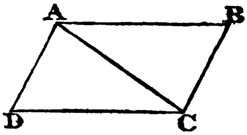
b Ax. 8.

PROP. XIV. THEOREM

The opposite sides and angles of any parallelogram ABCD are equal to one another, and the diameter BC bisects it, that is, divides it into two equal parts.

N.B.—*A parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diameter is the straight line joining two of its opposite angles.*

a 11. 1. Because AB is parallel to CD, and AC meets them, the alternate angles BAC, ACD, are equal to one another[a]; and because AD is parallel to BC, and AC meets them, the alternate angles ACB, CAD are also equal to one another[a]; wherefore, the two triangles ABC, ADC, have two angles BAC, ACB in the one, equal to two angles ACD, CAD in the other, each to each, and the side BC, which is adjacent to these equal angles, common to both; therefore, the triangles are equal in every respect[b]; b 13. 1. consequently, the side AB is equal to the side CD, and AD to BC, and the angle ABC equal to the angle ADC. And because the angle BAC is equal to the angle ACD, and the angle CAD to the angle ACB, the whole angle BAD is equal to the whole angle BCD; and the angle ABC has been shown equal to the angle ADC; therefore, the opposite sides and angles of a parallelogram are equal to one another. And since the triangle ABC is equal to the triangle ADC, the diameter of course bisects it. Therefore, &c. Q. E. D.



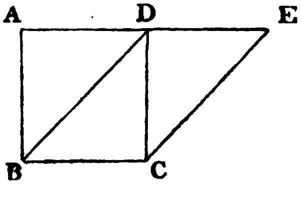
Cor. The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.

PROP. XV. THEOREM.

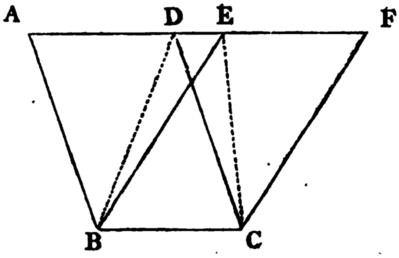
Parallelograms upon the same base, and between the same parallels, are equivalent.

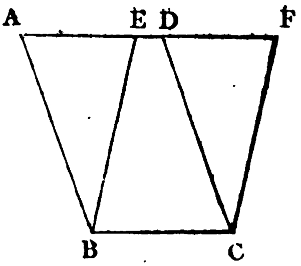
Let the parallelograms ABCD, EBCF, be upon the same base BC, and between the same parallels AF, BC, the parallelogram ABCD is equivalent to the parallelogram EBCF. See the 2d and 3d Figures.

If the sides AD, DF, of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D, it is plain, that each of the parallelograms is double of the triangle BDC[a], and they are therefore equivalent. a 14. 1.



But, if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point, then, because ABCD is a parallelogram, AD is equal to BC[a]; for the same reason, EF is equal to BC; wherefore, AD is equal to EF, and DE is common, therefore, the whole, or the remainder AE, is equal to the whole, or the remainder DF; now, AB is also equal to DC[a], therefore, the two EA, AB, are equal to the two FD, DC, each to each, but the exterior angle FDC is equal to the interior EAB[b], wherefore, the triangle EAB is equal to the triangle FDC[c]. Take the triangle FDC from the trapezium ABCF, and from the same trapezium take the triangle EAB, the remainders will then be equivalent, *that is*, the parallelogram ABCD, is equivalent to the parallelogram EBCF. Therefore, parallelograms, upon the same base, &c. Q. E. D.





a 14. 1

b 11. 1.

c 2. 1.

Cor. 1. If the diagonals BD and EC be drawn, then the triangles DBC and EBC are equivalent, being the halves of the parallelograms; hence triangles upon the same base, and between the same parallels, are equivalent.

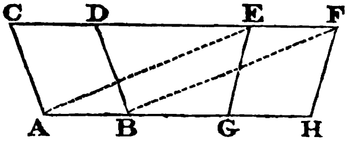
Cor. 2. If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.

Cor. 3. Equivalent triangles upon the same base, and upon the same side of it, are between the same parallels.

PROP. XVI. THEOREM.

Parallelograms, CABD, EFHG, upon equal bases, AB and GH, and between the same parallels CF and AH, are equivalent.

s Sup. Join AE, BF; and because AB is equal to GH[s], and a 14. 1. GH to EF[a], AB is equal to EF; and they are parallels, and joined towards the same parts by straight lines AE, BF; but straight lines which join equal and parallel straight lines towards the same parts, are themselves equal and parallel[b]; b Cor. 14. therefore, AE, BF are both equal and parallel, and EABF is a parallelogram, and it is equivalent to CABD, because it is upon the same base AB, and between the same parallels CF, AB[c]. c 51.1. For the like reason, the parallelogram EFHG is equivalent to the same EABF. Therefore, also the parallelogram CABD is equivalent to EFHG. Wherefore, parallelograms, &c. Q. E. D.



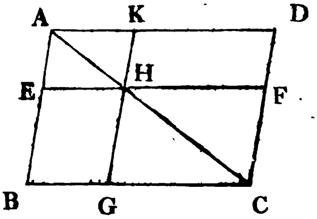
Cor. 1. Triangles upon equal bases, and between the same parallels, are equivalent.

Cor. 2. Equivalent triangles on the same side of bases which are equal, and in the same straight line, are between the same parallels.

PROP. XVII. THEOREM.

The complements BH and HD of the parallelograms EK and GF, which are about the diameter AC of any parallelogram, BD, are equivalent.

Because ABCD is a parallelogram, and AC its diameter, a 14. 1. the triangle ABC is equal to the triangle ADC[a]; and because EKHA is a parallelogram, and AH its diameter, the triangle AEH is equal to the triangle AHK. For the same reason, the triangle HGC is equal to the triangle HFC; the triangle AEH, together with the triangle HGC, is equal to the triangle AHK, together with the triangle HFC; but the whole triangle ABC, is equal to the whole triangle ADC; therefore, the remaining complement BH is equivalent to the remaining complement HD. Wherefore, the complements, &c. Q. E. D.

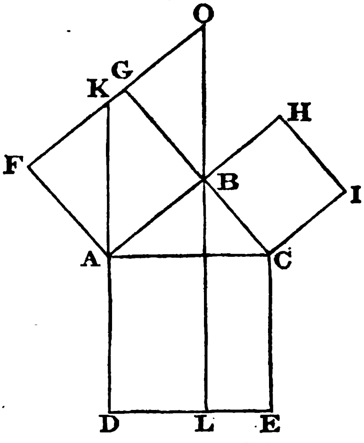


PROP. XVIII. THEOREM.

In any right-angled triangle, ABC, the square which is described upon the side AC, subtending the right angle ABC, is equivalent to the sum of the squares described upon the sides AB, BC, which contain the right angle.

Upon AC, AB, and BC, describe the squares AE, AG, and BI; produce DA to K; and through B draw LBO parallel to DK, meeting FG produced in O.

The angle CAK is equal to the angle FAB, each being a right angle; take away the common angle BAK, and the remaining angle CAB is equal to the remaining angle FAK; the angle ABC is also equal to the angle AFK, being right angles, and the side AB is equal to the corresponding side AF, because they are sides of a square; hence, the triangles ABC and AFK are equal in every respect[a]; and, therefore, AK is equal to AC or AD. Now the parallelogram AL is equivalent to the parallelogram AO, because they are upon equal bases AD and AK, and between



a 13.1.

b 16. 1.
c 15. 1.

the same parallels DK and LO[b]; and the square AG is also equivalent to the parallelogram AO, since they are upon the same base AB, and between the same parallels FO and AB[c]; therefore, the square AG is equivalent to the parallelogram AL. In like manner, by producing EC and IH, it may be demonstrated, that the square BI is equivalent to the parallelogram CL; wherefore, the whole square ACED is equivalent to the two squares AG and BI. Therefore, in any right-angled triangle, &c. Q. E. D.

Cor. If the square described upon one of the sides of a triangle, be equivalent to the sum of the squares described upon the other two sides, the angle contained by these two sides is a right angle.

ELEMENTS
OF
GEOMETRY.

BOOK II.

DEFINITIONS.

Book II.

I.—A STRAIGHT line is said to touch a circle, when it meets the circle, and being produced, does not cut it. The touching line is called a *tangent*.

II.—Circles are said to touch one another which meet, but do not cut one another.

III.—An arc of a circle is any part of the circumference; and the straight line joining its extremities is called a *chord*.

IV.—Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal; and the straight line on which the greater perpendicular falls, is said to be farther from the centre.

V.—A segment of a circle is the figure contained by a straight line and the arc which it cuts off.

VI.—An angle in a segment of a circle, is the angle contained by two straight lines, drawn from any point in the circumference of the segment, to the extremities of the straight line, which is the base of the segment.

VII.—And an angle is said to insist or stand upon the arch intercepted between the straight lines which contain the angle.

VIII.—The sector of a circle, is the figure contained by two radii, and the arc of the circumference between them.

IX.—When all the angular points of one rectilineal figure are placed in the sides of another, and the one wholly within the other, the former is said to be *inscribed* in the latter, or the latter *described* about the former.

X.—When all the angular points of a rectilineal figure are in the circumference of a circle, the figure is said to be *inscribed* in the circle, or the circle to be *described* about it.

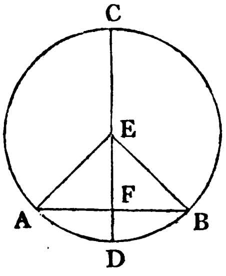
XI.—When the sides of a rectilineal figure are tangents to a circle, the figure is said to be *described* about the circle, or the circle to be *inscribed* in it.

PROP. I. THEOREM.

If a straight line CD, drawn through the centre E of a circle, bisect a straight line AB in the circle, which does not pass through the centre, it will cut it at right angles; and if it cut it at right angles, it will bisect it.

Join EA and EB; then, because EA is equal to EB[e], and AF to FB[s] and FE common to the two triangles, AFE and BFE, they are equal in every respect[a]. Hence, the angle AFE is equal to the angle BFE; and, therefore, each of them is a right angle[b]; wherefore, CD cuts AB at right angles.

e Cor. Def. 14. 1.
s Sup.
a 4. 1.
b Def. 7.



Again, let CD cut AB at right angles; CD also bisects AB. The same construction being made, because the radii EA and EB are equal, the angle EAF is equal to the angle EBF[c]; and the right angle EFA is equal to the right angle EFB, and the side EF is common to both; therefore, in the two triangles EAF and EBF, there are two angles and a side in the one equal to two angles and a corresponding side in the other; therefore, the triangles are equal in every respect[d]; AF is therefore equal to FB. Wherefore, if a straight line, &c. Q. E. D.

c 3. 1.
d 13. 1.

Cor. 1. If a chord be bisected at right angles by a straight line, that straight line will pass through the centre; and, hence, if it be produced both ways to the circumference, and then bisected, the point of bisection will be the centre of the circle.

Cor. 2. If two chords be bisected at right angles, the point of intersection of the bisecting lines, will be the centre of the circle.

Cor. 3. Hence a circle may be described about a triangle; for if two of its sides be considered as chords, and be bisected as in Cor. 2, the point where the bisecting lines meet, will be the centre of the circle.

Cor. 4. Hence, also, a circle may be completed, of which a segment is given; for the centre may be found as in the two last Cors.

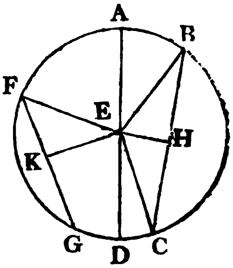
PROP. II. THEOREM.

The diameter AD is the greatest straight line in a circle; and of all others, BC, FG; BC, which is nearer to the centre, is always greater than FG, the one more remote, and the greater is nearer to the centre than the less.

From the centre E, draw EH, EK, perpendiculars to BC, FG[a], and join EB, EC, EF; and because AE is equal to EB, and ED to EC, AD is equal to EB, EC; but EB, EC, are greater than BC[b]; wherefore, AD is also greater than BC.

a Cor.1.4.1.

b Cor.Ax.12.



And because BC is nearer to the centre than FG, EH is less than EK, but BC is double of BH, and FG double of FK[c]. Now, the square of EB is equal to the square of EF, but the square of EB is equivalent to the squares of EH and HB[d], and the square of EF is equivalent to the squares of EK and KF[d]; therefore, the squares of EH, HB, are equivalent to the squares of EK, KF; of which the square of EH is less than the square of EK, consequently, the square of BH is greater than the square of FK, and the straight line BH greater than FK; and, therefore, BC is greater than FG. Next, let BC be greater than FG, BC is nearer to the centre than FG, that is, the same construction being made, EH is less than EK. Because BC is greater than FG, BH is likewise greater than KF; but the squares of BH, HE, are equivalent to the squares of FK, KE, of which the square of BH is greater than the square of FK, because BH is greater than FK; therefore, the square of EH is less than the square of EK, and the straight line EH less than FK. Wherefore, the diameter, &c. Q. E. D.

c 1. 2.

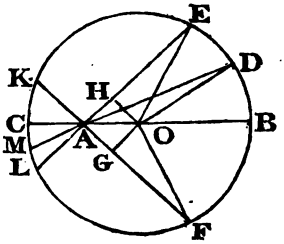
d 18. 1.

Cor. Equal straight lines in a circle are equally distant from the centre; and those which are equally distant from the centre, are equally distant from one another.

PROP. III. THEOREM.

Of all the straight lines that can be drawn to meet the circumference of a circle, from any eccentric point, that which passes through the centre is the greatest; that which, when produced, passes through the centre is the least; and that which is nearer to the centre is greater than the more remote.

Let O be the centre of the circle BCF, and A another point, from which the straight lines AB, AD, AE and AC are drawn to the circumference, AB passing through the centre is the greatest, AC the least, and AD is greater than AE.



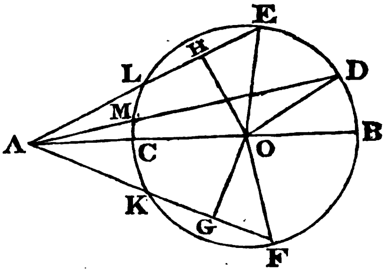
Join OD, OE; then, since OD is equal to OB, AO and OD are equal to AO and OB or AB; but OA and OD are greater than AD[a]; therefore, AB is also greater than AD. Again, the two triangles AEO and ADO, have the two sides AO, OE respectively equal to the two sides AO, OD, but the angle AOD is greater than the angle AOE[b]; therefore, the base AD is greater than the base AE[c]. Again, because OA and AM[d] are greater than OM*, and OM equal to OC, therefore, AC is less than AM; and, by joining OM and OL, it may be proved by Prop. VIII., Book I., that AM is less than AL. In like manner, it may be shown, when the point A is without the circle, that AC is less than AM, and AM less than AL.

a Cor. Ax. 12

b Ax. 9.

c 8. 1.

d Cor. Ax. 12.



Cor. 1. If, from any eccentric point, straight lines be drawn to the circumference, equally distant from the centre, they are equal. If AF and AE be drawn from A to the

* To preserve distinctness in the figure, the lines OL and OM are not actually drawn.

circumference, so that the perpendiculars OG and OH let fall upon them from the centre O, are equal, AF is equal to AE. It is obvious, that EH and GF are the halves of two equal chords, since they are equally distant from the centre, d Cor. 2. 2. and consequently are equal[d]. Farther, AOH and AOG being two right-angled triangles, having the same hypothenuse AO, and the equal sides OH and OG, it is easy to show (by the help of Prop. XVIII., Book I.) that AH is equal to AG; and hence AE is equal to AF

Cor. 2. From the same eccentric point, there cannot be drawn more than two equal straight lines to the circumference of a circle.

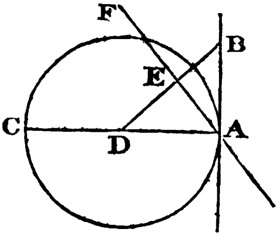
Cor. 3. That point from which three equal straight lines can be drawn to the circumference, is the centre of the circle.

Cor. 4. One circle cannot cut another in more than two points.

PROP. IV. THEOREM.

The perpendicular AB, at the extremity of a diameter AC, is a tangent to the circle; and no straight line can be drawn between the tangent and circumference, from the extremity of the diameter, so as not to cut the circle.

Let B be any point in the straight line AB, join this point and the centre D. Then, because DAB is a right angle, it is greater than the angle a Cor. 6. 1. DBA[a]; wherefore, DB is greater b Cor. 7. 1. than the radius DA[b], consequently, the point B is without the circle. Now, B is any point whatever in the line AB; therefore, AB falls without the circle.



Again, between the tangent AB, and the circumference, no straight line can be drawn from the point A which does not cut the circle. Let AF be drawn in the c Cor. 4. 1. angle DAB; from D draw DE at right angles to AF[c]; then, because the angle DEA is a right angle, DAE is less than a right angle[a]; therefore, DE is less than the radius

DA[b], consequently, the point E is within the circle; and, therefore, AF cuts the circle. Wherefore, the perpendicular, &c. Q. E. D. b Cor. 7. 1.

Cor. 1. At the same point, in the circumference of a circle, there cannot be more than one tangent.

Cor. 2. If a straight line touch a circle, the straight line drawn from the centre to the point of contact, is perpendicular to the line touching the circle.

Cor. 3. The perpendicular to a tangent from the point of contact, passes through the centre.

Scholium. The proposition suggests the method of drawing a tangent to the circle from any given point in the circumference. For, if the given point and the centre be joined, a straight line drawn at right angles to this line, through the given point, will obviously touch the circle.

◆

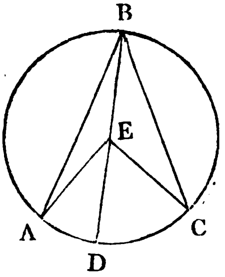
PROP. V. THEOREM.

The angle AEC, at the centre of a circle, is double of the angle ABC at the circumference, upon the same base AC, that is, upon the same part of the circumference.

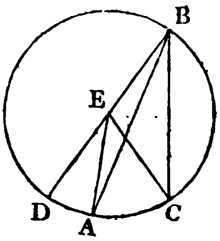
First, Let E, the centre of the circle, be within the angle ABC, join BE, and produce it to D. Because EB is equal to EA, the angle EBA is equal to the angle EAB[a]; therefore, the two angles EBA and EAB, are together double of the angle EBA; but the exterior angle AED is equal to the two interior and opposite angles EAB and EBA[b], therefore, the angle AED is also double of the angle ABE. For the same reason, the angle DEC is double of the angle EBC; therefore, the whole angle AEC is double of the whole angle ABC.

a 3. 1.

b 12. 1.



Again, let E, the centre of the circle, be without the angle ABC, join BE, and produce it to D. It may be demonstrated, as in the first case, that the angle DEC is double of the angle DBC, and that DEA, a part of the first, is double of DBA, a part of the other; therefore, the remaining angle AEC is double of the remaining angle ABC. Therefore, the angle at the centre, &c. Q. E. D.

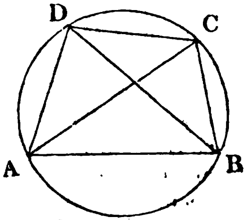


Cor. The angles at the circumference, which stand upon the same arc, are equal, or, which is the same thing, the angles in the same segment of a circle are equal, because they are the halves of the angle at the centre standing upon the same arc.

PROP. VI. THEOREM.

The opposite angles of any quadrilateral figure ABCD, inscribed in a circle, are together equal to two right angles.

Draw the diagonals AC and BD. The angle CAB is equal to the angle CDB[a], because they are in the same segment BADC, and the angle ACB is equal to the angle ADB[a], because they are in the same segment ADCB; therefore, the whole angle ADC is equal to the two angles CAB and ACB. To each of these equals add the angle ABC; and the angles ABC, ADC, are equal to the three angles ABC, BCA, and CAB. But ABC, BCA, and CAB, are together equal to two right angles[b]; therefore, the angles ABC and ADC are also together equal to two right angles. In the same manner, the angles BAD, DCB, may be shown to be together equal to two right angles. Therefore, the opposite angles, &c. Q. E. D.



a Cor. 5. 1.

b 12. 1.

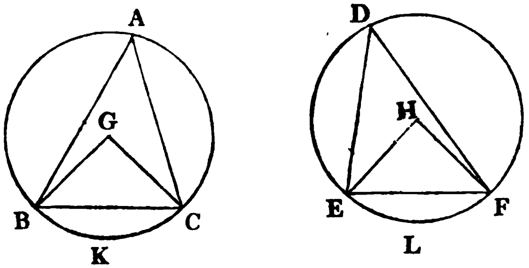
Cor. If one of the sides of a quadrilateral figure, inscribed in a circle, be produced, the exterior angle is equal to the interior and opposite angle.

PROP. VII. THEOREM.

In equal circles, equal angles stand upon equal arches, whether they be at the centres or circumferences.

Let ABC, DEF, be two equal circles, having the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences; the arch BKC is equal to the arch ELF.

Join BC, EF. Then, because the circles are equal, their radii are equal; therefore, the two sides BG, GC, are equal to the two EH, HF; and the angle at G is equal to the angle at H[s]; therefore, the base BC is equal to the base EF[a]. Conceive the circle ABC applied to the circle DEF, so that the points G, B, C, of the triangle GBC may coincide with the points H, E, F of the triangle HEF; then, since every point in the arch BKC is at the same distance from the centre G, that every point in the arch ELF is from the centre H, and the centres G and H coincide; therefore, the arches BKC and ELF also coincide, and, consequently, are equal. Wherefore, in equal circles, &c. Q. E. D.



s Sup.
a 2. 1.

Cor. 1. In equal circles, the angles which stand upon equal arches are equal, whether they be at the centres or circumferences.

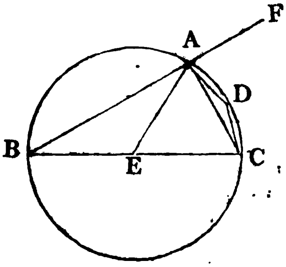
Cor. 2. In equal circles, equal straight lines cut off equal arches, the greater equal to the greater, and the less equal to the less. For, if BC be equal to EF, it follows, (by Prop. IV., Book I.) that the angles BGC and EHF are equal; and, hence, that the arches BKC and ELF are likewise equal;

and since the circles are equal, consequently the remaining arches BAC and EDF are also equal.

Cor. 3. In equal circles, equal arches are subtended by equal straight lines. For, since the arches BKC and ELF are equal; it follows from Cor. 1., that the angles BGC and EHF are equal; and, hence, that the bases BC and EF are
b 2. 1. also equal[b].

PROP. VIII. THEOREM.

In a circle, the angle BAC in a semicirle is a right angle; but, the angle ABC in a segment greater than a semicircle, is less than a right angle; and the angle ADC, in a segment, less than a semicirle, is greater than a right angle.



Produce BA to F, take E, the centre, and join AE. Then, because EA is equal to EB, the angle EAB is equal to the
a 3. 1. angle EBA;[a] also, because EA is equal to EC, the angle EAC, is equal to the angle ECA[a]; wherefore, the whole angle BAC is equal to the two angles ABC and ACB. But the exterior angle FAC is also equal to the two interior and opposite
b 12. 1. angles ABC and ACB[b]; therefore, the angle BAC is equal to the angle FAC, and they are adjacent angles, therefore each
c Def. 7. 1. of them is a right angle[c]; wherefore, the angle BAC in a semicircle is a right angle.

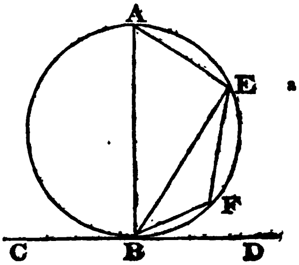
Again, because the two angles ABC, BAC, of the triangle
d Cor. 6. 1. ABC are together less than two right angles[d], and BAC is a right angle, ABC must be less than a right angle; and, therefore, the angle in a segment ABC, greater than a semicircle, is less than a right angle. Farther, because ABCD is a quadrilateral figure inscribed in a circle, any two if its opposite angles
e 6. 2. ABC, ADC are together equal to two right angles[e]; and ABC is less than a right angle; wherefore, the other ADC is greater than a right angle. Therefore, in a circle, &c. Q. E. D.

Cor. From this, it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal to the same two; and, when the adjacent angles are equal, they are right angles.

PROP. IX. THEOREM.

The angles which the chord, drawn from the point of contact, makes with the tangent, are equal to those in the alternate segments of the circle.

Let CD be a tangent to the circle at B, and BE any chord drawn from the point of contact B, dividing the circle into the two segments BAE and BFE, the angles EBD and EBC will be respectively equal to the angles EAB and EFB.



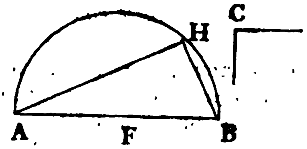
From the point B, draw BA perpendicular to CD, join AE; and let the lines BF and EF be drawn to any point F, in the arc BE. Because BA is drawn at right angles to the tangent CD, from the point of contact B, it is a diameter[a]; therefore, the angle AEB being in a semicircle, is a right angle[b]; and, consequently, the other two angles EBA and EAB, are together equal to a right angle[c]; but ABD is likewise a right angle; therefore, the angle ABD is equal to the two angles EBA and EAB; take from these equals the common angle EBA; and there will remain the angle EBD, equal to the angle EAB, which is in the alternate segment of the circle. Again, because AEFB is a quadrilateral figure inscribed in a circle, the opposite angles BAE and BFE, are together equal to two right angles[d]; therefore, the angles EBD and EBC, being also equal to two right angles[e], are together equal to the two BAE and BFE; of which the angle EBD has been shown to be equal to the angle BAE, therefore, the remaining angle EBC is equal to the remaining angle BFE in the alternate segment of the circle. Wherefore, the angle, &c. Q. E. D.

a Cor. 3. 4. 2.
b 8. 2.
c 12. 1.
d 6. 2.
e Cor. Def. 7. 1.

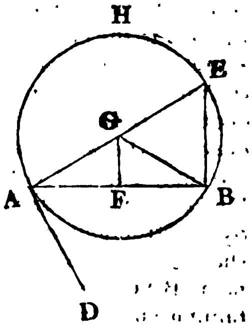
Cor. If a straight line meet the circumference of a circle, and make the angle with a chord at that point, equal to the angle in the alternate segment; it will be a tangent to the circle.

PROP. X. PROBLEM.

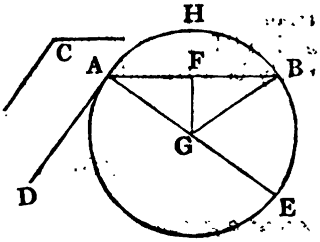
Upon a given straight line AB, to describe a segment of circle that shall contain an angle equal to a given rectilineal angle C.



First, let the angle at C be a right angle; bisect AB in a Cor. 2. 4.1. F[a], and from the centre F at the distance FB, describe the semicircle AHB; the angle AHB being in a semicircle, is equal to the right angle at C.



But, if the angle at C, be not a right angle at the point A, in the straight line AB, make the angle b Sch. 4. 1. BAD equal to the angle C[b], and from the point A, draw AE at right angles c Cor. 1. 4.1. to AD[c]; bisect AB in F, and from F draw FG at right angles to AB[c], and join GB. Then, because AF is equal to FB, and FG common, and the angle AFG equal to the angle BFG, the triangles AFG and BFG are equal d 2. 1. in every respect[d]; and, consequently, AG is equal to GB: and the circle described from the centre G, at the distance GA, will pass through the point B; let this be the circle AHB; and, because, from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, therefore, AD e 4. 2. touches the circle[e]: and because the angle which the chord makes with the tangent at the point of contact, is equal to the angle in the alternate segment f 9. 2. of the circle[f]; therefore, the angle BAD is equal to the angle in the alternate segment c Cons. AHB; but the angle BAD is equal to the angle C[c]; therefore, also the angle C is equal to the angle in the segment AHB. Wherefore, upon the given straight line AB, a

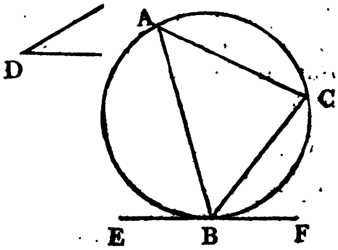


segment AHB of a circle has been described, containing an angle equal to the given angle C. Which was to be done.

PROP. XI. PROBLEM.

From a given circle ABC, to cut off a segment, that shall contain an angle equal to a given rectilineal angle D.

Take any point B, in the circle ABC, and through B draw the tangent EBF[a]; at the point B, in the straight line FB, make the angle FBC equal to the angle D[b]. Then, because EF is a tangent to the circle at the point B, and BC a chord drawn from the point of contact; the angle CBF is equal to the angle in the alternate segment BAC[c]; but the angle CBF, is, by construction, equal to the angle D, therefore, the angle in the segment BAC is also equal to D. Wherefore, the segment BAC has been cut off from the given circle ABC, containing an angle equal to the given angle D, which was to be done.



a Sch. 4. 2.

b Sch. 4. 1.

c 9. 2.

Scholium. This proposition suggests the method of inscribing, in a given circle, a triangle equiangular to a given triangle. For, if CBF be made equal to one of its angles, and ABE equal to another, and AC joined, ABC will manifestly be a triangle equiangular to the given triangle.

PROP. XII. PROBLEM.

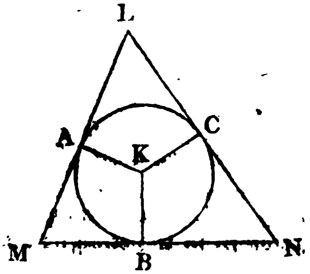
About a given circle ABC, to describe a triangle equiangular to a given triangle DEF.

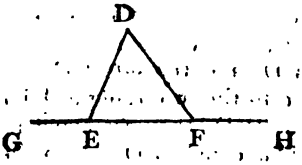
Produce EF both ways, to the points G and H. Find the centre K of the given circle ABC[a], and from it draw any

Book IV. radius KB; at the point K, make the angle BKA equal to

a Cor. 1.1.2. the angle DEG[b]; and the angle BKC equal to the angle

b Sch. 4. 1.



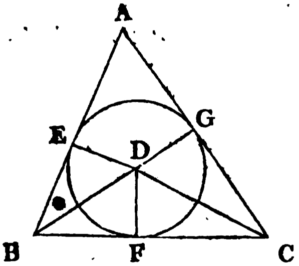


DFH[b]; and through the points A, B, C, draw the tangents g Sch. 4. 2. LAM, MBN, and NCL[g]. Then, because LM, MN, and NL are tangents, and AK, BK, CK are drawn from the points of contact to the centre, the angles at A, B, and C c Cor. 2. 4.2. are right angles[c]. And because the four angles of the quadrilateral figure AMBK, are together equal to four right d Cor. 1. 12. 1. angles[d]; and because two of them KAM and KBM are right angles, the other two AKB, AMB, are together equal to two right angles: But the angles DEF, DEG, are also together e Cor. Def. 7. 1. equal to two right angles[e]; therefore, the angles AKB, AMB, are equal to the angles DEG, DEF, of which AKB is equal to DEG; wherefore, the remaining angle AMB, or LMN is equal to the remaining angle DEF. In like manner, it may be shown, that the angle LNM is equal to the angle DFE; and, therefore, the remaining angle MLN is equal to f 12. 1. the remaining angle EDF[f]: Wherefore, the triangle LMN is equiangular to the triangle DEF; and it is described about the given circle ABC. Which was to be done.

PROP. XIII. PROBLEM

To inscribe a circle in a given triangle ABC.

Bisect each of the angles ABC, ACB, by the straight lines a Cor. 2.4.1. BD, CD[a], meeting one another in the point D, from which



draw DE, DF, DG perpendicular to AB, BC, CA respectively[b]. Then, because the angle EBD is equal to the angle FBD, the angle ABC being bisected by BD; and because the right angle BED is equal to the right angle BFD, and the side BD common to the two triangles BED and BFD, they are equal in every respect[c]; and, therefore, DE is equal to DF. In the same manner, it may be shown, that DG is equal to DF; therefore, the three straight lines DE, DF, and DG are equal to one another, and the circle described from the centre D at the distance of any one of them, will pass through the extremities of the other two, and will touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right angles: Therefore, a circle EFG has been inscribed in the given triangle ABC. Which was to be done.

[b] Cor. 1. 4. 1.

[c] 13. 1.

The following problems are so simple, that it is thought unnecessary to subjoin the figures:—

PROP. XIV. PROBLEM.

To inscribe a square in a given circle.

Draw two diameters at right angles to one another, and join their extremities.

PROP. XV. PROBLEM.

To describe a square about a given circle.

Through the extremities of two diameters at right angles, draw tangents.

PROP. XVI. PROBLEM.

To inscribe a circle in a given square.

The intersection of the diagonals is the centre, and the perpendicular from it on a side the radius.

PROP. XVII. PROBLEM.

To describe a circle about a given square.

The intersection of the diagonals is the centre, and its distance from one of the angular points the radius.

PROP. XVIII. PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle.

The radius of the circle is the side.

PROP. XIX. PROBLEM.

To describe an equilateral and equiangular hexagon about a given circle.

Divide the circumference into six equal parts, as in the preceding problem, and through the points of section draw tangents.

ELEMENTS

OF

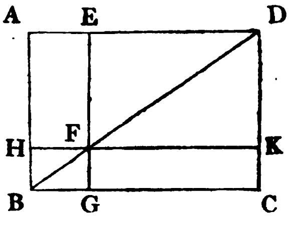
GEOMETRY.

BOOK III.

DEFINITIONS.

I.—EVERY right-angled parallelogram or *rectangle*, is said to be contained by any two of the straight lines which are about one of its angles. Thus, the right-angled parallelogram AC, is called the rectangle contained by AD and DC, or by AD and AB, &c. For the sake of brevity, instead of the rectangle *contained by* AD and DC, we shall simply say, the rectangle AD.DC, placing a point between the two sides of the rectangle. Also, instead of saying the *square* of any line, for instance of AD, we shall occasionally write AD^2.

II.—In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a *Gnomon*. Thus, the parallelogram HG, together with the complements AF, FC, or the parallelogram EK, with these same complements, are the gnomons of the parallelogram AC. These gnomons may also, for the sake of brevity, be called, the first AGK, or EHC, and the second, AKG or HEC.



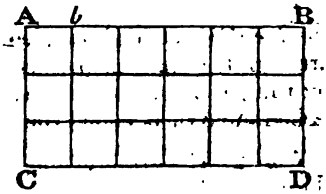
Book III.

Every quantity is measured by some other quantity of the same kind, of a known magnitude. Thus, a line is measured by a line, a surface by a surface, and a solid by a solid. The measuring unit in the case of lines, may be an inch, a foot, a yard, &c.; in the case of surfaces, a square inch, square foot, square yard, &c.; and, in the case of solids, a cubic inch, cubic foot, cubic yard, &c.

LEMMA.

The product of the two numbers which express the lengths of the adjacent sides of a rectangle, will express the area or quantity of surface contained in it, as measured by the square described upon the linear unit.

Let ABDC be a rectangle, and A*b* the linear unit, or unit of length; then, if AB contain this unit 6 times, and AC 3 times, the rectangle will contain a space equal to 18 times the square described upon A*b*.



If, through the points of section in AB and AC, parallels be drawn as in the figure, then it is manifest, that each of the small surfaces is equal to the square described on A*b*. Now, in the figure, there are as many squares in a row, as there are linear units in AB, and as many rows as there are linear units in AC; hence, the whole number of squares will be expressed by the product of the units in AB by those in AC. In the present case, the area will be equal to $6 \times 3 = 18$. The same might be shown to hold, although the sides may not be expressed by whole numbers. Thus, if the length of a rectangle were $8\frac{3}{4}$ inches, and its breadth $4\frac{1}{2}$ inches, then its area will be equal to $8\frac{3}{4} \times 4\frac{1}{2} = 39\frac{3}{8}$ square inches.

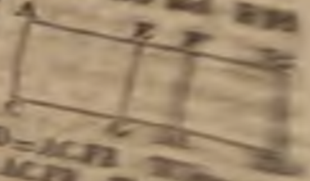
Cor. The area of a geometrical square is equal to the numerical square of its side. Hence the reason why the product of a number by itself is called the *square* of the number.

By means of the preceding lemma, the student will be enabled to illustrate the following propositions numerically.

PROP. I. THEOREM.

The rectangle contained by two straight lines AC, AB, is equivalent to the sum of the rectangles contained by the one line AC, and the several segments AE, EF, FB, into which the other may be divided.

Or, $AC.AB = AC.AE + AC.EF + AC.FB$.



Through the points of section E, F, draw EG and FH parallel to AC or BD. The figures AG, EH, and FD are evidently rectangles; and AD = AG+EH+FD. But AD = AC.AB; AG = AC.AE; EH = EG.EF = AC.EF, because EG = AC; in like manner, FD = AC.FB. Therefore, AC.AB = AC.AE + AC.EF + AC.FB. Therefore the rectangle, &c. Q. E. D.

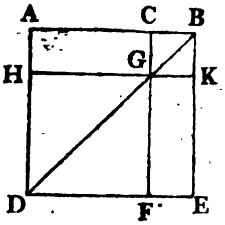
Cor. 1. If a straight line [illegible] the [illegible] are together [illegible] by the [illegible] by the Prop. [illegible] AC = AB, [illegible]

Cor. 2. If a straight line [illegible] the rectangle [illegible] is equivalent [illegible] gether with [illegible] AC.AF = [illegible] AC.AF = [illegible]

PROP. II. THEOREM.

If a straight line [illegible] *CB, the* [illegible]

Upon AB [illegible] through C [illegible]



G, draw HK parallel to AB or DE. The exterior angle CGB is equal to
a 11. 1. the interior angle ADB[a], and, there-
b 3. 1. fore, also to CBG[b], consequently, CB
c Cor. 1. 3. 1. is equal to CG[c]; but CB is equal to
d 14. 1. GK, and CG to BK[d], wherefore, the parallelogram CBKG is equilateral. It is likewise rectangular, for the angle CBK being a right angle, it follows that its other angles are also right
e 11. 1. angles[c]; wherefore, CBKG is a square, and it is upon CB. For the same reason, HF is also a square, and it is upon HG, which is equal to AC[d]; therefore, HF and CK are the squares of AC and CB. The complements AG and GE are
f 17. 1. equivalent[f], but $AG = AC.CG = AC.CB$; therefore, also $GE = AC.CB$, and $AG + GE = 2AC.CB$. Now, $HF = AC^2$, and $CK = CB^2$; therefore, $HF + CK + AG + GE = AC^2 + CB^2 + 2AC.CB$. But $HF + CK + AG + GE =$ the whole $AE = AB^2$; therefore, $AB^2 = AC^2 + CB^2 + 2AC.CB$. Wherefore, if a straight line, &c. Q. E. D.

Cor. 1. From this demonstration, it is manifest that the parallelograms about the diameter of a square are likewise squares.

Cor. 2. The square of any line is equivalent to four times the square of half the line.

Cor. 3. If a straight line AB, be divided into any two parts, AC, CB, the squares of the whole line, and of one of the parts, are equivalent to twice the rectangle contained by the whole, and that part, together with the square of the other part. By the Prop. $AB^2 = AC^2 + CB^2 + 2AC.CB$, add CB^2 to each; then $AB^2 + CB^2 = AC^2 + 2CB^2 + 2AC.CB$; but, $AC.CB + CB^2 = AG + CK = AK = AB.BK = AB.BC$; and, therefore, twice $2AC.CB + 2BC^2 = 2AB.BC$. Hence, $AB^2 + BC^2 = 2AB.BC + AC^2$.

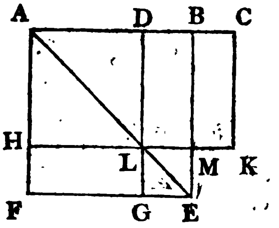
Cor. 4. Hence the sum of the squares of any two lines is equivalent to twice the rectangle contained by the lines, together with the square of the difference of the lines.

PROP. III. THEOREM.

The rectangle contained by the sum and difference of two straight lines AB, and BC is equivalent to the difference of their squares.

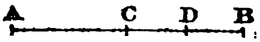
$$\text{Or, } (AB+BC).(AB-BC) = AB^2 - BC^2.$$

From BA, cut off BD, equal to BC. Then is AC the sum of the two lines AB and BC, and AD their difference. On AB, describe the square ABEF, and join AE; through D, draw DG parallel to AF or BE; also through L draw HLMK parallel to AB or FE, meeting CK, drawn parallel to AF or BE, in the point K. Then, because BC is equal to BD, and BD to LM,[a] therefore, BC is equal to LM, and, consequently, GM is equal to BC^2[b]. Again, because BK is equal to BL[c], and BL equal to LF[d], being complements, therefore BK is equal to LF; to each add AM; then AK is equal to the gnomon GAM. But $AK = AC.AH = AC.AD$, (because $AH = AD$ being sides of a square) $= (AB+BC).(AB - BC)$. And the gnomon $GAM = FB - GM = AB^2 - BC^2$; therefore, $(AB+BC).(AB-BC) = AB^2 - BC^2$. Wherefore, the rectangle, &c. Q. E. D.



a 14. 1.
b Cor.1.2.3.
c 16. 1.
d 17. 1.

Cor. 1. If a straight line AB be divided into two equal parts AB, CE, and also into two unequal parts AD, DB; the rectangle AD.DB, contained by the unequal segments, together with the square of CD, the line between the points of section is equivalent to the square of AC or CB half the line. For AD is the sum of AC, CD, and DB is their difference. Now, by the Prop. $AD.DB = AC^2 - CD^2$, and consequently $AD.DB + CD^2 = AC^2$.



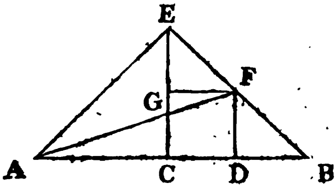
Cor. 2. If a straight line AB be bisected in C, and produced to any point D, the rectangle AD.DB contained by the whole line thus produced, and the part produced, together with the square of CB, half the line bisected, is equivalent to

Book II. square of CD, the line made up of the half, and the part produced. For AD is the sum of AC, and CD, and DB is their difference, and hence, as before, by the Prop. $AD.DB = CD^2 - CB^2$, or $AD.DB + CB^2 = CD^2$.

PROP. IV. THEOREM.

If a straight line AB be divided into two equal parts AC, CB, and also into two unequal parts AD, DB; the squares of the two unequal parts are together double of the square of half the line, and of the square of the line CD, between the points of section.

$$\text{Or, } AD^2 + DB^2 = 2AC^2 + 2CD^2.$$

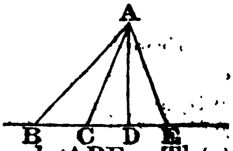


From the point C, draw CE at right angles to AB, and make it equal to AC or CB, and join AE, EB; through D, draw DF parallel to CE, and through F, draw FG parallel to AB, and join AF. Then, because the angle ECA is a
a 12. 1. right angle, the two others
b 2. 1. CEA and CAE are together equal to a right angle[a]; and they are equal to one another, because CE is equal to CA[b]; therefore, each is half a right angle: For the same reason, each of the angles CEB and CBE is half a right angle; therefore, the whole angle AEB is a right angle: And because GEF is half a right angle, and EGF a right angle, GFE is also half
c Cor. 1. 3. a right angle[a], and, therefore, equal to GEF; consequently, GE is equal to GF[c]. In the same manner, it may be shown,
d 18. 1. that DF is equal to DB. Now, $AE^2 = AC^2 + CE^2$[d] $= 2AC^2$; and $EF^2 = EG^2 + GF^2 = 2GF^2 = 2CD^2$, because $CD = GF$; therefore, $AE^2 + EF^2 = 2AC^2 + 2CD^2$; but $AE^2 + EF^2 = AF^2$[d]; therefore, $AF^2 = 2AC^2 + 2CD^2$. Again, $AF^2 = AD^2 + DF^2$[d] $= AD^2 + DB^2$, because $DB = DF$; therefore, $AD^2 + DB^2 = 2AC^2 + 2CD^2$. Therefore, if a straight line, &c. Q. E. D.

PROP. V. THEOREM.

The square of one of the sides of a triangle is greater or less than the sum of the squares of the other two sides, by twice the rectangle contained by the base and its segment, intercepted between the perpendicular and the angle opposite to that side, according as that opposite angle is obtuse or acute.

First, let ABC be an obtuse angled-triangle, having the obtuse angle ACB; let AD be drawn perpendicular to BC produced.* Then AB^2 is *greater* than BC^2+AC^2 by 2BC.CD. Because BD is divided into any two parts in the point C, $BD^2=BC^2+CD^2+2BC.CD$[a]; add AD^2 to both; then $BD^2+AD^2=BC^2+CD^2+AD^2+2BC.CD$. a 2. 3.



But $BD^2+AD^2=AB^2$[b], and $CD^2+AD^2=AC^2$[b], therefore, $AB^2=BC^2+AC^2+2BC.CD$; *that is*, AB^2 is greater than BC^2+AC^2, by 2BC.CD. Again, let AEB be an acute angle of the triangle ABE. Then, in this case, AB^2 is *less* than BE^2+EA^2 by twice the rectangle BE.ED. Because BE is divided into any two parts in the point D, $BE^2+ED^2=2BE.ED+BD^2$[c]; add AD^2 to both; then $BE^2+ED^2+AD^2=2BE.ED+BD^2+AD^2$; but $ED^2+AD^2=EA^2$[b]; and $BD^2+AD^2=AB^2$; therefore $BE^2+EA^2=AB^2+2BE.ED$; or AB^2 is *less* than BE^2+AE^2 by 2BE.ED.

b 18. 1.

c Cor. 3.2. 3.

Cor. 1.—If a straight line AC be drawn from the vertex A of any triangle ABD, so as to bisect the base, BD, the sum of the squares of the other two sides AB, AD, is double of the square of BC half the side bisected, and of the square of AC the bisecting line. For, as is shown in the proposition $AB^2=BC^2+AC^2+2BC.CD=2BC^2+AC^2+CD^2$ (because BC = CD;) add AD^2 to both; then $AB^2+AD^2=2BC^2+AC^2+CD^2+AD^2=2BC^2+2AC^2$, because $CD^2+AD^2=AC^2$[b]. b 18. 1.

Cor. 2. The sum of the squares of the sides of any parallelogram, is equivalent to the sum of the squares of its diameters.

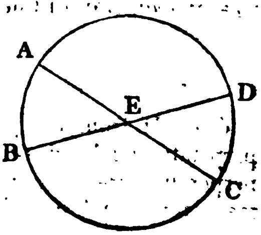
Cor. 3. Any angle of a triangle is obtuse, right, or acute, according as the square of the opposite side is greater than, equal to, or less than the sum of the square of these sides which contain it.

---

* If ACB be an obtuse angle, it is easy to show that the perpendicular AD will fall without the triangle ABC. And if AEB be an acute angle, then AD will fall within the triangle ABE.

PROP. VI. THEOREM.

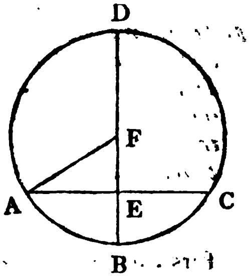
If two straight lines AC, BD, within a circle, cut one another, the rectangle AE.EC contained by the segments of the one, is equivalent to the rectangle BE.ED contained by the segments of the other.



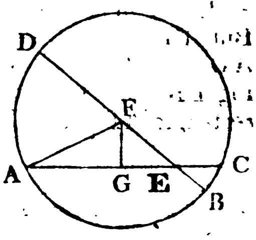
If AC, BD, pass each of them through the centre E; then it is evident that AE, EC, BE, and ED being all equal, the rectangle AE.EC = BE.ED.

But let one of them BD, pass through the centre, and cut the other AC, which does not pass through the centre at right angles in the point E: then if BD be bisected in F, F is the centre[a]; join AF. Then because BD passes through the centre, and cuts AC, which does not pass through the centre at right angles, it also bisects it[b]; therefore, AE is equal to EC. And because BD is cut into two equal parts in the point F, and into two unequal parts in the point E, $BE.ED+EF^2=FB^2$ [c] $=AF^2$. But $AF^2=AE^2+EF^2$ [d]; therefore, $BE.ED+EF^2=AE^2+EF^2$, and taking EF^2 from each, there remains $BE.ED=AE^2=AE.EC$.

a Cor. 1. 1.2.
b. 1. 2
c Cor. 1.3.3.
d 18. 1.



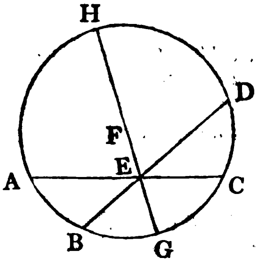
Next, let BD pass through the centre, and cut AC, which does not pass through the centre in E, but not at right angles: then, as before, if BD be bisected in F, F is the centre; join AF, and from F, draw FG perpendicular to AC; therefore AG is equal to GC[b]; wherefore $AE.EC+EG^2=AG^2$ [c], and adding CF^2 to both $AE.EC+EG^2+GF^2=AG^2+GF^2$. Now $EG^2+GF^2=EF^2$ [d]; and $AG^2+GF^2=AF^2=BF^2$; therefore, $AE.EC+EF^2=BF^2$; but



$BF^2 = BE.ED + EF^2$[c]; therefore, $AE.EC + EF^2 = BE.ED + EF^2$; and taking EF^2 from both, there remains the rectangle $AE.EC = BE.ED$.

c Cor. 1. 3.3.

Lastly, let neither of the straight lines AC, BD, pass through the centre F. Through the point E, the intersection of AC. BD, draw the diamater HFEG. Then, by the last case, $AE, EC = GE.EH$, and $BE.ED = GE.EH$; therefore, $AE.EC = BE.ED$. Wherefore, if two straight lines &c. Q,ED.

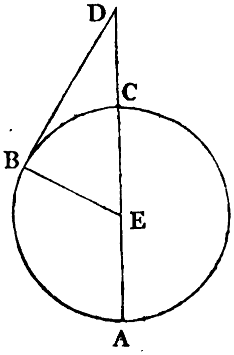


Cor. If, from any point in the circumference of a circle, a perpendicular be drawn to the diameter, the square of this perpendicular, is equivalent to the rectangle under the segments into which it divides the diameter.

PROP. VII. THEOREM.

If from any point D, without a circle, two straight lines be drawn, one of which DA cuts the circle, and the other DB touches it; the rectangle AD.DC contained by the whole line which cuts the circle, and the part of it without the circle, is equal to the square of DB, the line which touches it.

First, let DCA pass through the centre E, join EB; then the angle EBD is a right angle[a]; and because the straight line AC is bisected in E, and produced to D, $AD.DC + EC^2 = ED^2$[b], or $AD.DC + EB^2 = ED^2$. But $ED^2 = EB^2 + BD^2$[c]; therefore $AD.DC + EB^2 = EB^2 + BD^2$, and taking EB^2 from both, there remains the rectangle $AD.DC = DB^2$.

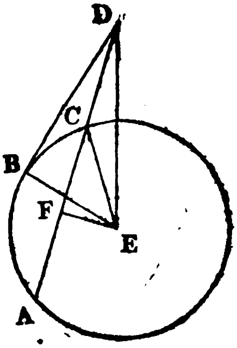


a Cor. 2.4.2.

b Cor. 2.3.3.

c 18. 1.

Again, suppose that DCA does not pass through the centre E. Draw EF perpendcular to AC, then is AF d 1. 2. equal to FC[d]; join EB, EC, and FD. Then, as before, $AD.DC+CF^2 = DF^{2}$[b], and adding FE^2 to both $AD.DC+CF^2+FE^2 = DF^2+FE^2$. But $CF^2+FE^2 = CE^2 = BE^2$; and $DF^2+FE^2 = DE^2 = DB^2+BE^2$[c], because the angle DBE is a right angle[a]; therefore, $AD.DC+BE^2 = DB^2+BE^2$, and, taking BE^2 from both, there remains the rectangle $AD.DC = DB^2$. Wherefore, if from any point, &c. Q. E. D.



Cor. 1. If, from any point without a circle, there be drawn two straight lines cutting it, the rectangles contained by the whole lines, and the parts of them without the circle, are equivalent.

Cor. 2. If from any point without a circle, there be drawn two tangents to it, these tangents are equal.

Cor. 3. If from a point without a circle, there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equivalent to the square of the line which meets it, the line which meets also touches the circle.

ELEMENTS

OF

GEOMETRY.

BOOK IV.

DEFINITIONS.

Book IV.

I.—SIMILAR rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

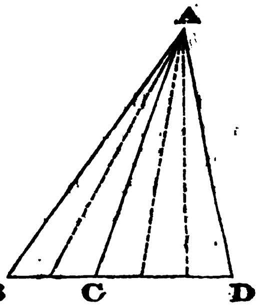
II.—Two sides of one figure are said to be reciprocally proportional to two sides of another, when one of the sides of the first is to one of the sides of the second, as the remaining side of the second is to the remaining side of the first.

III.—The altitude of any figure is the straight line drawn from its vertex perpendicular to its base.

PROP. I. THEOREM.

Triangles of the same altitude are to one another as their bases.

Let the triangles ABC, ACD, have the same altitude, viz. the perpendicular drawn from the point A to BD. Then the triangle ABC is to the triangle ACD, as the base BC is to the base CD.



Case 1. When the bases are commensurable.

If BC : CD :: m : n, then △ ABC : △ ACD also as m : n.

For BC being divided into as many equal parts as there are units in m, BD will contain as many such equal parts as there are units in n; the division being made, and straight lines drawn from A to the several points of section, it is manifest that the whole triangle ABD will be divided into $m+n$, equal
a Cor.1.16.1. triangles[a]. Now, the △ ABC will contain as many of these equal triangles as there are units in m, and ACD as many of them as there are units in n. Hence, △ ABC : △ ACD : : m
b 111. Alg. : n; but BC : CD by supposition as m : n, and therefore[b] △ ABC : △ ACD :: BC : CD.

Case 2. When the bases are incommensurable.

From the demonstration of Case I., it is obvious, that if BC be greater than $\frac{m}{n}$ of CD, but less than $\frac{m+1}{n}$ of CD, then the △ ABC will be greater than $\frac{m}{n}$ of △ ACD, but less than $\frac{m+1}{n}$ of ACD, however great the magnitudes m and n may
c 124. Alg. be; and hence[c], that the △ ABC : △ ACD :: BC : CD.— Wherefore triangles, &c. Q. E. D.

Cor. 1. Parallelograms of the same altitude are to one another as their bases.

Cor 2. Triangles and parallelograms on the same base are to one another as their altitudes.

PROP. II. THEOREM.

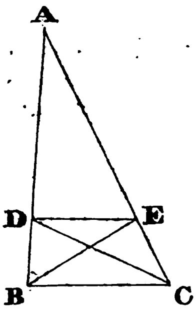
If a straight line DE be drawn parallel to one of the sides BC, of a triangle ABC, it will cut the other sides, or the other sides produced proportionally; and if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section, will be parallel to the remaining side of the triangle.

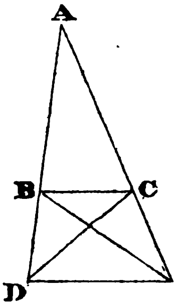
Join BE and CD; then the triangle BDE is equivalent to the triangle CDE[a], and therefore the triangle BDE : triangle ADE :: triangle CDE : triangle ADE*. But the triangle BDE : triangle ADE :: BD : DA[b]; and the triangle CDE : triangle ADE :: CE : EA. Hence BD : DA :: CE : EA[c]

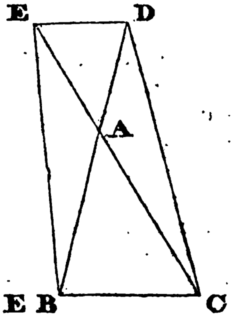
a Cor. 1. 15. 1.

b 1. 4.

c 111. Alg.







Again, if BD : DA :: CE : EA, then is DE parallel to BC. The same construction being made, we have, by supposition, BD : DA :: CE : EA, but BD : DA :: triangle BDE : triangle ADE, and CE : EA :: triangle CDE : triangle ADE[b], therefore the triangle BDE : triangle ADE :: triangle CDE : triangle ADE[c], *that is*, the triangles BDE and CDE have the same ratio to ADE, and, therefore, BDE and CDE are equivalent; and they are upon the same base DE, hence they are between the same parallels[d], there-

d Cor. 3. 15. 1.

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* It is hardly necessary to *prove*, that equal magnitudes have to the same the same ratio; or that magnitudes which have to the same the same ratio, are equal to one another. If *a* and *b* be two equal quantities, and *c* a third quantity, then $\frac{a}{c} = \frac{b}{c}$, and hence $a : c :: b : c$. Conversely, if $a : c :: b : c$, then $\frac{a}{c} = \frac{b}{c}$, and hence $a = b$.

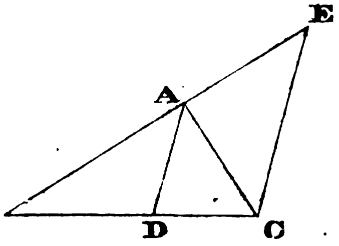
fore DE is parallel to BC. Wherefore, if a straight line, &c. Q. E. D.

PROP. III. THEOREM.

If the vertical angle BAC of a triangle ABC, be bisected by a straight line AD, which also cuts the base, the segments BD, DC of the base shall have the same ratio, which the other sides BA, AC of the triangle have to one another: And if the segments of the base have the same ratio which the other two sides of the triangle have to one another, the straight line, drawn from the vertex to the point of section, bisects the vertical angle.

Through C, draw CE parallel to AD, meeting BA produced in E. The exterior angle BAD is equal to the interior
b 11. 1. and opposite angle AEC[b], and the alternate angles DAC and ACE are likewise equal[b]; but the angle BAD is, by supposition, equal to DAC, therefore, the angle AEC is
c Cor. 1. 3. 1. equal to the angle ACE, and hence AE is equal to AC[c]. And because AD is parallel to EC, a side of the triangle
d 2. 4. BCE, BA : AE :: BD : DC[d]; but AE has been shown to be equal to AC, therefore BA : AC :: BD : DC.

Next, if BA : AC :: BD : DC, and AD be joined, the angle BAC is bisected by the straight line AD. The same construction being made; because AD is parallel to EC, BA : AE :: BD : DC[d], but BA : AC :: BD : DC by supposition; hence
e 111. Alg. BA : AE :: BA : AC[e], that is, AE and AC have the same ratio to BA, and therefore AE is equal to AC, and, conse-
f 3. 1. quently, the angle AEC is equal to the angle ACE[f]. Now, BAD is equal to the angle AEC, and DAC to ACE[b]; wherefore, also, the angle BAD is equal to the angle DAC, that is, the angle BAC is bisected by BD. Wherefore, if the vertical angle, &c. Q. E. D.

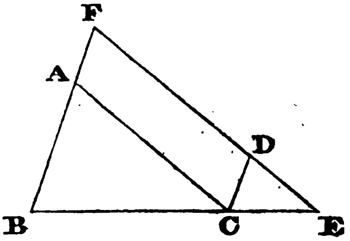


PROP. IV. THEOREM.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, on the antecedents or consequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, ACB equal to DEC, and, consequently[a], BAC equal to CDE, then will the sides about these equal angles be proportionals. a 12. 1.

Let the triangle DCE be so placed, that its sides CE may be contiguous to BC, and in the same straight line with it. And because the angles ABC, ACB, are together less than two right angles[b], ABC and DEC, which is equal to ACB, are also less than two right angles, wherefore, BA, ED produced shall meet[c]; let them be produced and meet in the point F, and because the angle DCE is equal to the angle ABC[s], DC is parallel to FB[d]. Again, because the angle ACB is equal to FEB[s], AC is parallel to FD[d]; therefore, AFDC is a parallelogram, and, consequently, AF is equal to CD, and AC to FD[e]. Now because AC is parallel to FE, a side of the triangle BFE BA : AF : : BC : CE[f], but AF is equal to CD, therefore, BA : CD : : BC : : CE, and alternately BA : BC : : CD : CE[g]. Again, because CD is parallel to BF, BC : CE : : FD : DE, but FD is equal to AC, therefore, BC : CE : : AC : DE, and alternately BC : CA : : CE : ED[g]. Lastly, since it has been proved that BA : BC : : CD: CE, and BC : CA : : CE : ED, by multiplying the corresponding terms of these analogies together, we have BA × BC : BC × CA : : CD × CE : CE × ED[m], and, therefore, by dividing the first two terms of the analogy by BC, and the last two by CE, BA : AC : : CD : DE[h]. Wherefore, the sides, &c. Q. E. D.

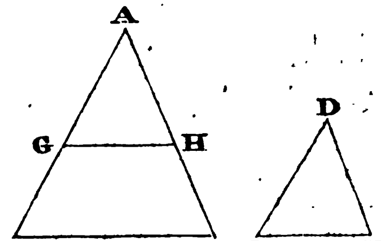


b Cor. 6. 1.
c Cor. 1. 11. 1.
s Sup.
d 10. 1.
e 14. 1.
f 2. 4.
g 112. Alg.
m 119. Alg.
h 118. Alg.

PROP. V. THEOREM.

If the sides of two triangles about each of these angles be proportionals, the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.

Let the triangles ABC, DEF, have their sides proportionals, so that AB : BC : : DE : EF, and BC : CA : : EF : FD, and, consequently, from composition and simplification as in the last proposition, BA : AC : : ED : DF, the triangle ABC is equiangular to the triangle DEF. From AB cut off AG equal to DE, and draw GH parallel to BC. Then the triangules ABC and AGH, having the angle at A common, and AGH equal to ABC, and AHG to ACB are equiangular. Then AB : BC : : AG : GH[a], but AB : BC : DE : : EF by supposition, therefore, AG : GH : : DE : EF[b], but AG is equal to DE[c] ; therefore, AG : GH : : DE : EF[b], but AG is equal to DE[c], therefore, GH equal to EF. In the same manner, it may be proved, that AH is equal to DF ; hence the two triangles AGH, and DEF having the three sides of the one respectively equal to the three sides of the other are equal in every respect[a] ; and therefore, the angle at A equal to the angle at D, the angle AGH, that is, ABC equal to DEF ; and, consequently, the third angle ACB equal to DFE. Wherefore, if the sides, &c. Q. E. D.

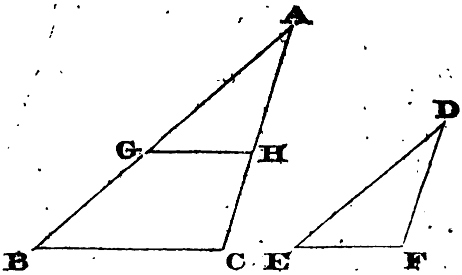


a 4. 4.
b 111. Alg.
c Cons.
b 111. Alg.
c Cors.
a 4. 1.

PROP. VI. THEOREM.

If two triangles have one angle in the one, equal to one angle in the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Let the triangles ABC, DEF, have the angle BAC in the one, equal to the angle EDF in the other, and the sides about those equal angles proportionals, that is, BA : AC : :



ED : DF; the triangles are equiangular. From AB and AC, cut off AG equal to DE, and AH equal to DF, and join GH. Then, because GA is equal to ED, and AH to DF, and the included angle GAH equal to the included angle EDF[s], the triangles AGH and DEF are equal in every respect[a]; the angle AGH equal to DEF, and AHG to DFE. Again, because BA : AC :: ED : DF[s], and AG, AH respectively equal to DE, DF, BA : AC :: GA : AH, alternately BA : GA :: AC : AH[b], and, therefore, *dividendo*[c] BG : GA :: CH : HA; hence GH is parallel to BC[d], consequently, the angle AGH, that is, DEF, is equal to ABC[e]; and, therefore, also the third angle ACB equa to DFE. Wherefore, if two triangles, &c. Q. E. D.

s Sup.
a 2. 1.
b 112. Alg.
c 11. Alg.
d 2. 4.
e 11. 1.

PROP. VII. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then, if each of the remaining angles be either less, or not less, than a right angle, the triangles shall be equiangular, and have those angles equal, about which the sides are proportionals.

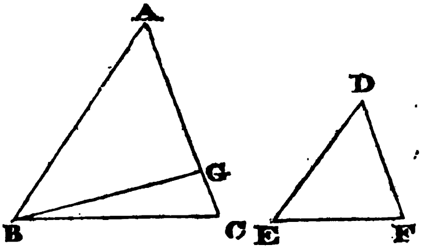
Let the two triangles ABC, DEF have one angle BAC of the one, equal to one angle EDF of the other, and the sides about two other angles ABC DEF proportionals, so that AB : BC :: DE : EF; and, in the first case, let each of the remaining angles at C, F, be less than a right angle; the triangles are equiangular.

For, if the angles ABC, DEF be not equal, one of them is greater than the other; let ABC be the greater, and, at the point B in the straight line AB, make the angle ABG equal to the angle DEF. Then, because the angle at A is equal to the angle at D, and ABG equal to DEF, the third angle AGB[a] is equal to the third angle DFE; and, therefore, the triangles ABG and DEF are equiangular; wherefore, AB : BG :: DE : EF[b], but DE : EF :: AB : BC by

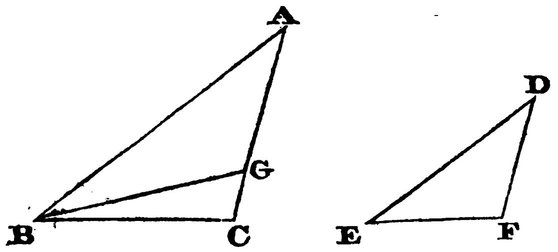
a 12. 1.
b 4. 4.

supposition; hence, AB : BG : AB : BC[c]; and, therefore, BG is equal to BC; and, consequently, the angle BGC[d] equal to BCG. But BCG is by hypothesis less than a right angle, therefore, BGC is also less than a right angle, and the adjacent angle AGB[e] must be greater than a right angle[e]. But AGB was proved to be equal to the angle at F, therefore, the angle at F is greater than a right angle. But, by supposition, it is less than a right angle, which is absurd. Therefore, the angles ABC, DEF are not unequal, that is, they are equal. And the angle at A is equal to the angle at D; wherefore, the remaining angle at C is equal to the remaining angle at F[a]; therefore, the triangle ABC is equiangular to the triangle DEF.

c 111. Alg.
d 3. 1.
e Cor. Def. 7.



Next, let each of the angles at C, F, be not less than a right angle; the triangle ABC is also, in this case, equiangular to the triangle DEF. The same construction being made, it may be proved in like manner that BG is equal to BC, and the angle at C equal to the angle BGC. But the angle at C is not less than a right angle; therefore, the angle BGC is not less than a right angle: Wherefore, two angles of the triangle BGC are together not less than two right angles, which is impossible[f]: And hence, as in the first case, the triangles ABC and DEF may be proved to be equiangular. Wherefore, if two triangles, &c. Q. E. D.

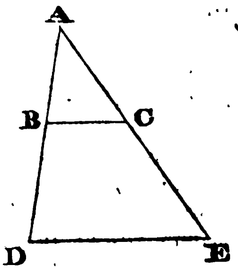


f Cor. 6. 1.

PROP. VIII. PROBLEM.

To find a third proportional to two given straight lines.

Let AB, AC, be the two given straight lines, and let them be placed so as to contain any angle BAC. Produce AB to D, making BD equal to AC; through D, draw DE parallel to BC, meeting AC produced in E. Then, because BC is parallel to DE, a side of the triangle ADE, AB : BD : : AC : CE[a]; but, BD = AC; therefore, AB : AC : : AC : CE. Wherefore, to the two given lines AB, AC, a third proportional CE has been found. Which was to be done.



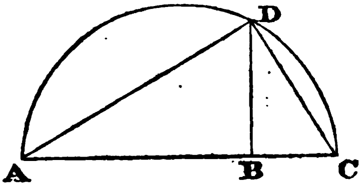
a 2. 4.

Cor. Hence a fourth proportional to three given straight lines may be found. For, if AB, BD and AC be the three given straight lines, then DE being drawn parallel BC, meeting AC produced in E, gives the fourth proportional CE.

PROP. IX. PROBLEM.

To find a mean proportional between two given straight lines AB, BC.

Place AB, BC in a straight line, and upon AC describe a semicircle ADC; draw BD at right angles to AC, and join AD, DC. Then the angle ADC being in a semicircle is a right angle[a], and, therefore, equal to ABD; now, the angle at A is common, hence the two triangles ABD and ACD are equiangular[b]. In the same way, may the triangle DBC be shown to be equiangular to ACD, therefore, the two triangles ABD and DBC are equiangular, and, consequently, have the sides about their equal angles proportionals, hence, AB : BD : : BD : BC[c]; that is BD is a mean proportional between AB and BC. Which was to be found.



a 8. 2.

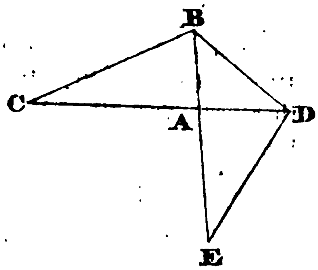
b 12. 2.

c 4. 4.

Cor. If, in a right-angled triangle, a perpendicular be drawn from the right angle to the base, it is a mean proportional between the segments of the base; the triangles on each side of it are similar to the whole and to one another, and each side is a mean proportional between the base and the segment adjacent to that side.

PROP. X. THEOREM.

Equivalent triangles ABC, ADE, which have one angle BAC of the one, equal to one angle DAE of the other; have their sides about the equal angles reciprocally proportional; and triangles which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equivalent.



Let the triangles be so placed that their sides CA, AD,
may be in the same straight line;
then, it is easy to show that EA,
AB, will also be in the same straight
line; join BD. Because the tri-
s Sup. angles are equivalent[s], they have
the same ratio to the triangle
ABD; that is the triangle CAB:
triangle BAD : : triangle EAD :
triangle BAD; but CAB :
a 1. 4. BAD : : CA : AD[a], and EAD :
b 111. Alg. BAD : : EA : AB[a]; therefore, CA : AD : : EA : AB[b].

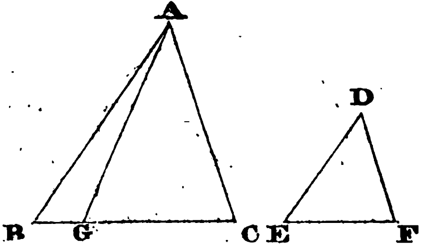
Again, if the angle CAB be equal to EAD and CA : AD : : EA : AB, the triangles shall be equivalent. Having joined BD as before. Because, CA : AD : : EA : AB[s] : and since CA : AD : : triangle CAB : triangle BAD[a]; and also, EA : AB : : triangle EAD : triangle BAD[a]; therefore, the triangle CAB : triangle BAD : : triangle EAD : triangle BAD[b]; that is, the triangles CAB and EAD have the same ratio to BAD, and they are therefore equivalent. Therefore, equivalent triangles, &c. Q. E. D.

Cor. The same may be demonstrated of parallelograms.

PROP. XI. THEOREM.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF, be similar triangles, having the angle B equal to the angle E, and let AB : BC : : DE : EF, so that BC is homologous to EF; then the triangle ABC : triangle DEF : : BC² : EF². Take BG a third proportional to BC
a 9. 6. and EF[a], so that BC : EF : : EF : BG, and join AG



Then, because AB : BC : : DE : EF, alternately AB : DE : : BC : EF[b]; but BC : EF : : EF : BG[c], therefore, AB : DE : : EF : BG[d], that is, the sides about the equal angles ABG and DEF of the triangles ABG and DEF are reciprocally proportional, therefore, the triangle ABG is equivalent to the triangle DEF[e]. Again, because BC : EF :: EF : BG, and that, if three quantities be in continued proportion, the first is to the third as the square of the first to the square of the second[f]; therefore, BC : BG : : $BC^2 : EF^2$; but, BC : BG : : triangle ABC : triangle ABG[g], therefore, triangle ABC : triangle ABG : : $BC^2 : EF^2$; but the triangle DEF is equivalent to the triangle ABG[c]; hence, triangle ABC : triangle DEF : : $BC^2 : EF^2$. Therefore, similar triangles, &c. Q. E. D.

b 112. Alg.
c Cons.
d 111. Alg.
e 10. 4.
f 121. Alg.
g 1. 4.

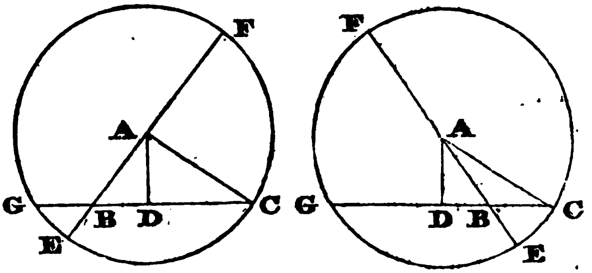
Cor. 1. From what is proved in the preceding proposition, it is easy to show that similar polygons are to one another as the squares of their corresponding sides.

Cor. 2. The rectilineal figure described upon the hypothenuse of a right-angled triangle, is equivalent to the sum of the similar rectilineal figures described upon the sides containing the right angle.

PROP. XII. THEOREM.

If, from any angle BAC of a triangle ABC, a perpendicular AD be drawn to the opposite side or base, the rectangle (AC + AB) (AC — AB) contained by the sum and difference of the other two sides is equivalent to the rectangle (CD + DB) (CD — DB) contained by the sum and difference of the segments into which the base is divided by the perpendicular.

From A as a centre, with the radius AC, the greater of the two sides describe the circle CFG; produce AB to meet the circumference in E and F, and CB to meet it in G. Then, because AF = AC, BF = AB + AC, the sum of the sides; and, since AE = AC, BE = AC — AB, the difference of the sides. Again, because AD, drawn from the



a 1. 2. centre, cuts GC at right angles, it also bisects it[a]; therefore, when the perpendicular falls within the triangle $BG = DG - DB = DC - DB$, the difference of the segments of the base; and $BC = BD + DC$ the sum of the segments. But when AD falls without the triangle $BG = DG + DB = CD + DB$, the sum of the segments of the base, and $BC = CD - DB$, the difference of the segments of the base. Now, in both cases, because B is the intersection of the two lines FE, GC drawn in the circle, $FB.BE = CB.BG$, that is, as has been shown, $(AC+AB).(AC-AB) = (CD+DB).(CD-DB)$. Therefore, &c. Q. E. D.

EXERCISES ON GEOMETRY.

BOOK I.

1. If a straight line bisect another at right angles, every point of the first line will be equally distant from the two extremities of the second line.

2. If two sides of a triangle be bisected at right angles, and from the point, where the bisecting lines cut one another, a perpendicular be drawn to the third side, it will bisect the third side.

3. If two angles of a triangle be bisected, and from the point where the bisecting lines cut one another, a straight line be drawn to the third angle, it will bisect the third angle.

4. Of all the straight lines that can be drawn from a point to a straight line, the perpendicular is the least; that which is nearer the perpendicular is less than the more remote; there can be two equal lines, but there cannot be more than two.

5. The difference of any two sides of a triangle is less than the third side.

6. If, from the vertex of a triangle, two straight lines be drawn, one of which is perpendicular to the base, and the other bisects the vertical angle, the angle contained by these lines shall be equal to half the difference of the angles at the base.

7. If a straight line bisect the diameter of a parallelogram, it will bisect the parallelogram.

8. If a straight line halve one side of a triangle, and be parallel to the base, it will halve the other side, and be equal to half the base; and the triangle cut off will be equal to one-fourth part of the original triangle.

9. If a straight line halve the two sides of an angle, it will be parallel to the base.

10. If a point be taken in each side of a square, at the same distance from the extremity, the figure formed by joining the four points will also be a square.

11. If, from the vertex of a triangle, two straight lines be drawn, cutting off from the base segments equal to the sides next them respectively, they shall contain an angle equal to half the sum of the angles at the base.

12. To construct a triangle, having the angles at the base equal to two given angles, (which are together less than the

right angles,) each to each, and its perimeter, or the sum of all its sides, equal to a given straight line.

BOOK II.

1. If, in a circle, a straight line bisect two parallel chords, it will pass through the centre.

2. Parallel chords intercept equal arcs.

3. Chords that intercept equal arcs, are parallel.

4. If two chords intersect at a point within the circle, the angle which they make is equal to an angle at the centre, standing on half the sum of the intercepted arcs.

5. If two chords intersect when produced, the angle which they make is equal to an angle at the centre, on half the difference of the intercepted arcs.

6. If two chords intersect at right angles, the sum of the two opposite arcs is equal to half the circumference.

7. A straight line drawn through the middle of an arc, parallel to its chord, is a tangent.

8. If two chords in a circle intersect at right angles, the sum of the squares of the four segments, will be equivalent to the square of the diameter.

9. If the circumference of a circle be divided into three equal parts, and from the three points straight lines be drawn to a fourth point in the circumference, one of the straight lines will be equal to the sum of the other two.

BOOK III.

1. If, from the vertex of an isosceles triangle, a straight line be drawn to meet the base, the square of this line, together with the rectangle under the segments of the base, shall be equal to the square of one of the equal sides of the triangle.

2. The square of the base of an isosceles triangle, is equal to twice the rectangle contained by either side, and by the straight line intercepted between the perpendicular, let fall upon it from the opposite angle and the extremity of the base.

3. If the vertical angle of a triangle be two-thirds of a right angle, the square of the base will be equal to the difference between the rectangle under the two sides and the sum of their squares.

4. In any quadrilateral figure, the sum of the squares of the diagonals, together with four times the square of the line joining their middle points, is equal to the sum of the squares of all the sides.

5. In any quadrilateral figure, the sum of the squares of the diagonals, is equal to double the sum of the squares of the two lines, that join the middle points of its opposite sides.

BOOK IV.

1. If two straight lines be cut by parallel lines, they are cut in the same ratio.

2. If, on two bases, there be placed the same number of triangles, similar each to each, and similarly situated, the polygons formed by joining the vertices of the triangles will be similar.

3. A straight line, drawn from the vertex of a triangle to meet the base, divides a parallel to the base in the same ratio as the base.

4. If, from two angular points of a triangle, straight lines be drawn to bisect the opposite sides, they will divide each other into segments, having the ratio of two to one.

5. From a given point, either within or without a given angle, to draw a straight line cutting the two lines which contain the angle, so that the distances of the two intersections from the given point shall be to one another in a given ratio.

6. To describe a right-angled triangle that shall have a given hypothenuse, and its three sides continually proportional.

THE END.

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